Rinehart MATHEMATICAL TABLES, FORMULAS AND CURVES

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Enlarged Edition

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1 Greek Alphabet

Letters		*Names*	*Letters*		*Names*
Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	o	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ϵ	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ s	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	ϕ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

2 Weights and Measures

LENGTH

12 inches = 1 foot
3 feet = 1 yard
$16\frac{1}{2}$ feet, $5\frac{1}{2}$ yards = 1 rod

63,360 inches, 5,280 feet, 1,760 yards, 320 rods = 1 mile

AREA

144 square inches = 1 square foot
9 square feet = 1 square yard
$30\frac{1}{4}$ square yards = 1 square rod
160 square rods = 1 acre
640 acres = 1 square mile = 1 section

VOLUME

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard

DRY MEASURE

2 pints = 1 quart
8 quarts = 1 peck
4 pecks = 1 bushel
1 bushel = 2150.42 cubic inches

LIQUID MEASURE

2 gills = 1 cup
2 cups = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon
$31\frac{1}{2}$ gallons = 1 barrel
2 barrels = 1 hogshead
1 gallon = 231 cubic inches

AVOIRDUPOIS WEIGHT

7000 grains = 16 ounces
16 ounces = 1 pound
2000 pounds = 1 ton
2240 pounds = 1 long ton

TROY WEIGHT

24 grains = 1 pennyweight
20 pennyweights = 1 ounce
12 ounces, 5760 grains = 1 pound

APOTHECARIES' WEIGHT

20 grains = 1 scruple
3 scruples = 1 dram
8 drams = 1 ounce
12 ounces = 1 pound

APOTHECARIES' LIQUID

60 minims = 1 fluid dram
8 fluid drams = 1 fluid ounce
16 fluid ounces = 1 pint
8 pints = 1 gallon

SURVEYOR'S LENGTH

7.92 inches = 1 link
25 links = 1 rod
4 rods = 1 chain
80 chains = 1 mile

SURVEYOR'S AREA

625 square links = 1 square rod
16 square rods = 1 square chain
10 square chains = 1 acre
640 acres = 1 square mile
1 square mile = 1 section
36 sections = 1 township

PAPER

24 sheets = 1 quire
20 quires = 1 ream
2 reams = 1 bundle
5 bundles = 1 bale
(500 sheets is commonly called a ream)

TEMPERATURE

Freezing point = 0° C. or 32° F.
Boiling point = 100° C. or 212° F.

$C = \frac{5}{9}(F - 32)$
$F = \frac{9}{5}C + 32$

MARITIME MEASURE

6 feet = 1 fathom
120 fathoms = 1 cable length
1 nautical mile = 6080.20 feet (U.S.)
3 nautical miles = 1 league
1 knot = 1 nautical mile per hour
1 displacement ton = 35 cubic feet
1 freight ton = 40 cubic feet
1 capacity ton = 100 cubic feet closed-in space

MISCELLANEOUS

1 acre = 40 yd. × 121 yd.
1 carat = 200 milligrams
1 cord = 128 cubic feet
1 hand = 4 inches
1 furlong = 40 rods
1 cubic foot = about $7\frac{1}{2}$ gallons
1 British bushel = 1.03205 U. S. bu.
1 British gallon = 1.20095 U. S. gal.

METRIC SYSTEM

The metric weights and measures are derived by combining the following six numerical prefixes with the words *meter*, *gram*, and *liter:*

milli = one-thousandth	*deka* = ten
centi = one-hundredth	*hecto* = one hundred
deci = one-tenth	*kilo* = one thousand

For example,

10 millimeters = 1 centimeter	10 meters = 1 dekameter
10 centimeters = 1 decimeter	10 dekameters = 1 hectometer
10 decimeters = 1 meter	10 hectometers = 1 kilometer

APPROXIMATE METRIC EQUIVALENTS

1 inch = 2.54001 centimeters
1 foot = 30.48006 centimeters
1 yard = 91.4402 centimeters
1 centimeter = 0.032808 feet
1 mile = 1.60935 kilometers
1 square mile = 2,589,998 square meters
1 acre = 4046.873 square meters
1 meter = 39.37 inches, exactly

1 kilometer = 0.62137 mile = 1093.61 yards = 3,280.83 feet
1 kilogram = 2.20462 pounds = 35.27396 ounces (Av.)
1 liter = 1.05671 liquid quarts = 0.908102 dry quart

1 pound (Av.) = 0.453592 kilograms
1 gallon = 3.78533 liters
1 liquid quart = 0.946333 liter
1 liter = 61.0250 cubic inches
1 cubic inch = 16.3872 cubic centimeters
1 cubic meter = 35.314 cubic feet
1 dry quart = 1.10120 liters

1 kilogram per square meter = 0.204817 pound per square foot
1 pound per square foot = 4.88241 kilograms per square meter
1 fluid ounce = 29.573 cubic centimeters

3 Miscellaneous Physical Constants

g (average value) = 32.16 ft./sec.^{2} = 980 cm./sec.^{2}

g (sea level, lat. 45°) = 32.172 ft./sec.^{2} = 980.616 cm./sec.^{2}

1 hp. = 550 ft.-lb./sec. = 33,000 ft.-lb./min. = 76.0404 kg.-m./sec. = 745.70 watts

Weight of 1 cu. ft. water = 62.425 lb. (max. density)

Velocity of light in a vacuum = 2.99776×10^3 m./sec. = 186,284 mi./sec.

Velocity of sound in dry air at 0° C. = 33,136 cm./sec. = 1,087 ft./sec.

1 mi./hr. = 88 ft./min. = 1.467 ft./sec. = 0.8684 knot

1 knot = 101.3 ft./min. = 1.689 ft./sec. = 1.152 mi./hr.

1 micron = 10^{-4} cm. 1 angstrom unit = 10^{-8} cm.

Mean radius of earth = 3,959 mi. = 6,371 km.

Equatorial diameter of earth = 7,926.68 mi. = 12,756.78 km.

Polar diameter of earth = 7,899.98 mi. = 12,713.82 km.

Constant of gravitation = 6.670×10^{-8} dyne · cm^2/gram^2

Electronic charge = 4.803×10^{-10} e.s.u.

Mass of electron = 9.107×10^{-28} gram

Mass of hydrogen atom = 1.673×10^{-24} gram

Avogadro's number = 6.023×10^{23} mole^{-1}

Planck's constant = 6.624×10^{-27} erg. sec.

4 Important Mathematical Constants

$$\pi = 3.14159\ 26535\ 89793$$
$$\log_{10}\pi = 0.49714\ 98726\ 94134$$
$$e = 2.71828\ 18284\ 59045$$
$$\log_{10} e = 0.43429\ 44819\ 03252$$
$$\log_e 10 = 2.30258\ 50929\ 94046$$
$$\log_{10}\log_{10} e = 9.63778\ 43113\ 00537$$
$$\log_e 2 = 0.69314\ 71805\ 59945$$

NUMBERS CONTAINING π

	N	$\log_{10} N$		N	$\log_{10} N$
π	3.141 5927	0.497 1499	π^2	9.869 6044	0.994 2997
2π	6.283 1853	0.798 1799	$2\pi^2$	19.739 2088	1.295 3297
3π	9.424 7780	0.974 2711	$4\pi^2$	39.478 4176	1.596 3597
4π	12.566 3706	1.099 2099	$\pi^2/2$	4.934 8022	0.693 2697
$\pi/2$	1.570 7963	0.196 1199	$1/\pi^2$	0.101 3212	9.005 7003 − 10
$3\pi/2$	4.712 3890	0.673 2411	$1/2\pi^2$	0.050 6606	8.704 6703 − 10
$\pi/3$	1.047 1976	0.020 0286	$1/4\pi^2$	0.025 3303	8.403 6403 − 10
$2\pi/3$	2.094 3951	0.321 0586	π^3	31.006 2767	1.491 4496
$4\pi/3$	4.188 7902	0.622 0886	$1/\pi^3$	0.032 2515	8.508 5504 − 10
$\pi/4$	0.785 3982	9.895 0899 − 10	$\sqrt{\pi}$	1.772 4539	0.248 5749
$3\pi/4$	2.356 1945	0.372 2111	$\sqrt{2\pi}$	2.506 6283	0.399 0899
$\pi/6$	0.523 5988	9.718 9986 − 10	$\frac{1}{2}\sqrt{\pi}$	0.886 2269	9.947 5449 − 10
$1/\pi$	0.318 3099	9.502 8501 − 10	$\sqrt{\pi/2}$	1.253 3141	0.098 0599
$2/\pi$	0.636 6198	9.803 8801 − 10	$1/\sqrt{\pi}$	0.564 1896	9.751 4251 − 10
$3/\pi$	0.954 9297	9.979 9714 − 10	$1/\sqrt{2\pi}$	0.398 9423	9.600 9101 − 10
$4/\pi$	1.273 2395	0.104 9101	$\sqrt{2/\pi}$	0.797 8846	9.901 9401 − 10
$1/2\pi$	0.159 1549	9.201 8201 − 10	$\sqrt[3]{\pi}$	1.464 5919	0.165 7166
$1/3\pi$	0.106 1033	9.025 7289 − 10	$\sqrt[3]{\pi/6}$	0.805 9960	9.906 3329 − 10
$1/4\pi$	0.079 5775	8.900 7901 − 10	$\sqrt[3]{\pi^2}$	2.145 0294	0.331 4332
$\pi/180$	0.017 4533	8.241 8774 − 10	$1/\sqrt[3]{\pi}$	0.682 7841	9.834 2834 − 10
$180/\pi$	57.295 7795	1.758 1226	$\sqrt[3]{3/4\pi}$	0.620 3505	9 792 6371 − 10

5 Formulas from Algebra

1. SPECIAL PRODUCTS AND FACTORS

$$(x+y)(x-y) = x^2 - y^2$$
$$(ax+by)(cx+dy) = acx^2 + (ad+bc)xy + bdy^2$$
$$(x+y)^2 = x^2 + 2xy + y^2 \qquad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x-y)^2 = x^2 - 2xy + y^2 \qquad (x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x+y+z+\cdots)^2 = x^2 + y^2 + z^2 + \cdots + 2(xy + xz + yz + \cdots)$$
$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$
$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$$
$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$$
$$x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - y^{n-1}) \quad (n \text{ even})$$
$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}) \quad (n \text{ odd})$$

2. PROPORTION

If $a : b = c : d$ or $\dfrac{a}{b} = \dfrac{c}{d}$, then

$$ad = bc \qquad \frac{a}{c} = \frac{b}{d} \qquad \frac{b}{a} = \frac{d}{c} \qquad \frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a-b}{b} = \frac{c-d}{d} \qquad \frac{a+b}{a-b} = \frac{c+d}{c-d} \qquad \frac{pa+qb}{ra+sb} = \frac{pc+qd}{rc+sd}.$$

If $a : m = m : b$, then $m = \sqrt{ab}$ is the mean proportional between a and b.

If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \cdots = k$, then

$$k = \frac{a+c+e+\cdots}{b+d+f+\cdots} = \frac{pa+qc+re+\cdots}{pb+qd+rf+\cdots}.$$

3. VARIATION

If y varies directly as x, then $y = kx$. (k = constant)

If y varies inversely as x, then $y = \frac{k}{x}$.

If y varies jointly as x and z, then $y = kxz$.

If y varies directly as x and inversely as z, then $y = \frac{kx}{z}$.

4. COMPLEX NUMBERS

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.

$i^{4p} = 1$, $i^{4p+1} = i$, $i^{4p+2} = -1$, $i^{4p+3} = -i$. (p an integer)

If $a + bi = 0$, then $a = b = 0$, and conversely.
If $a + bi = c + di$, then $a = c$, $b = d$, and conversely.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Polar form: $a + bi = r(\cos\alpha + i\sin\alpha)$,
where $r = \sqrt{a^2 + b^2}$, $\tan\alpha = \frac{b}{a}$.

Let $A = r(\cos\alpha + i\sin\alpha)$, $B = R(\cos\beta + i\sin\beta)$. Then

$$A \cdot B = rR[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$$

$$\frac{A}{B} = \frac{r}{R}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)]$$

$$A^n = r^n[\cos n\alpha + i\sin n\alpha]$$

$$A^{1/n} = r^{1/n}\left[\cos\frac{\alpha + k360^\circ}{n} + i\sin\frac{\alpha + k360^\circ}{n}\right], k = 0, 1, 2, \cdots, n - 1$$

5. RADICALS

If $R^q = A$, then R is a qth root of A. The *principal qth root* is denoted by $\sqrt[q]{A}$.

$$\sqrt[q]{A} \text{ is } \begin{cases} \text{positive if } A \text{ is positive} \\ \text{negative if } A \text{ is negative and } q \text{ odd} \\ \text{imaginary if } A \text{ is negative and } q \text{ even} \end{cases}$$

$$\sqrt[q]{a^q} = \begin{cases} a \text{ if } a \geqq 0 \\ a \text{ if } a < 0 \text{ and } q \text{ odd} \\ -a \text{ if } a < 0 \text{ and } q \text{ even} \end{cases}$$

$$\sqrt[q]{a} \cdot \sqrt[q]{b} = \sqrt[q]{ab}$$

$$\sqrt[q]{\frac{a}{b}} = \frac{\sqrt[q]{a}}{\sqrt[q]{b}}$$

(Except for $a < 0$, $b < 0$, and q even. For this case, use the rules given above for complex numbers.)

6. EXPONENTS

If p is a positive integer, $a^p = a \cdot a \cdot a \cdots$ to p factors.

$$a^0 = 1 \quad (a \neq 0) \qquad a^{-n} = \frac{1}{a^n} \qquad a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

$$a^m \cdot a^n = a^{m+n} \qquad (ab)^n = a^n b^n$$

$$a^m \div a^n = a^{m-n} \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^m)^n = a^{mn}$$

7. LOGARITHMS

Let M, N, b be positive and $b \neq 1$. Then

$$\log_b M \cdot N = \log_b M + \log_b N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b M^k = k \log_b M$$

$$\log_b \sqrt[q]{M} = \frac{1}{q}\log_b M$$

$$\log_b b = 1 \qquad \log_b 1 = 0 \qquad b^{\log_b M} = M$$

$$\log_a M = \log_b M \cdot \log_a b = \frac{\log_b M}{\log_b a}$$

$$\log_e M = 2.3026 \log_{10} M \qquad \log_{10} M = 0.43429 \log_e M$$

Rule for the characteristics of common logarithms: *The characteristic of the logarithm of* N *is numerically equal to the number of digits between the reference position of the decimal point and its actual position in* N. *It is positive or negative according as the decimal point is to the right or left of the reference position. (The reference position of a decimal point is immediately to the right of the first nonzero digit in the number.)*

8. QUADRATIC EQUATIONS

Let r_1, r_2 be the roots of $ax^2 + bx + c = 0 \quad (a \neq 0)$. Then

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad r_1 + r_2 = -\frac{b}{a} \qquad r_1 r_2 = \frac{c}{a}.$$

If a, b, c are real and

$b^2 - 4ac > 0$, the roots are real and unequal;

$b^2 - 4ac = 0$, the roots are real and equal;

$b^2 - 4ac < 0$, the roots are imaginary and unequal.

If a, b, c are rational and

$b^2 - 4ac = a$ perfect square, the roots are rational;

$b^2 - 4ac \neq a$ perfect square, the roots are irrational.

9. FACTORIALS

$$n! = \underline{|n} = 1 \cdot 2 \cdot 3 \cdots n \qquad n! = n \cdot (n-1)!$$

Stirling's approximation: $\quad n! = \sqrt{2\pi n}\, n^n e^{-n}$

Forsyth's approximation: $\quad n! = \sqrt{2\pi} \left\{ \frac{\sqrt{n^2 + n + \frac{1}{6}}}{e} \right\}^{n+\frac{1}{2}}$

$$\log_e n! = (n + \tfrac{1}{2}) \log_e n + \log_e \sqrt{2\pi} - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{2352n^5} - \cdots$$

$0! = 1$	$7! = 5{,}040$	$14! = 87{,}178{,}291{,}200$
$1! = 1$	$8! = 40{,}320$	$15! = 1{,}307{,}674{,}368{,}000$
$2! = 2$	$9! = 362{,}880$	$16! = 20{,}922{,}789{,}888{,}000$
$3! = 6$	$10! = 3{,}628{,}800$	$17! = 355{,}687{,}428{,}096{,}000$
$4! = 24$	$11! = 39{,}916{,}800$	$18! = 6{,}402{,}373{,}705{,}728{,}000$
$5! = 120$	$12! = 479{,}001{,}600$	$19! = 121{,}645{,}100{,}408{,}832{,}000$
$6! = 720$	$13! = 6{,}227{,}020{,}800$	$20! = 2{,}432{,}902{,}008{,}176{,}640{,}000$

10. THE BINOMIAL THEOREM

If n is a positive integer,

$$(x + y)^n = x^n + {}_nC_1 x^{n-1} y + {}_nC_2 x^{n-2} y^2 + \cdots + {}_nC_n y^n$$

where $\quad {}_nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$

is equal to the number of combinations of n distinct objects taking them r at a time.

$${}_nC_0 = {}_nC_n = 1 \qquad {}_nC_r = {}_nC_{n-r}.$$

n	${}_nC_0$	${}_nC_1$	${}_nC_2$	${}_nC_3$	${}_nC_4$	${}_nC_5$	${}_nC_6$	${}_nC_7$	${}_nC_8$	${}_nC_9$	${}_nC_{10}$
1	1	1	..	...	...	...	...	...	..	..	.
2	1	2	1	...	...	...	...	...	..	..	.
3	1	3	3	1	...	...	...	...	..	..	.
4	1	4	6	4	1	...	...	...	..	..	.
5	1	5	10	10	5	1	...	...	..	..	.
6	1	6	15	20	15	6	1	...	..	..	.
7	1	7	21	35	35	21	7	1	..	..	.
8	1	8	28	56	70	56	28	8	1	..	.
9	1	9	36	84	126	126	84	36	9	1	.
10	1	10	45	120	210	252	210	120	45	10	1

Note in the preceding table that each number, plus the number on its left, is equal to the number next below.

$${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n$$

$${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots + (-1)^n {}_nC_n = 0$$

$${}_nC_0 + {}_{n+1}C_1 + {}_{n+2}C_2 + \cdots + {}_{n+r}C_r = {}_{n+r+1}C_r$$

$$({}_nC_0)^2 + ({}_nC_1)^2 + ({}_nC_2)^2 + \cdots + ({}_nC_n)^2 = {}_{2n}C_n$$

$${}_xC_n + {}_xC_{n-1} \cdot {}_yC_1 + {}_xC_{n-2} \cdot {}_yC_2 + \cdots + {}_yC_n = {}_{x+y}C_n$$

11. PROGRESSIONS

Let $l =$ the nth term, $S =$ the sum of n terms.

Arithmetical progression: $a, a + d, a + 2d, \cdots$

$$l = a + (n-1)d \qquad S = \frac{n}{2}(a + l) = \frac{n}{2}\{2a + (n-1)d\}$$

Geometrical progression: $a, ar, ar^2, \cdots$

$$l = ar^{n-1} \qquad S = a\frac{r^n - 1}{r - 1} = \frac{rl - a}{r - 1}$$

If $r^2 < 1$, $\qquad \lim\limits_{n \to \infty} S = \dfrac{a}{1 - r}.$

12. INTERPOLATION

Let $U_a, U_b, U_c, \cdots$ denote the values of the function U_x for $x = a, b, c, \cdots$, and let

$$\Delta U_x = \frac{U_{x+h} - U_x}{h}, \quad \Delta^2 U_x = \Delta(\Delta U_x) = \frac{U_{x+2h} - 2U_{x+h} + U_x}{h^2}, \text{ etc.}$$

Lagrange's interpolation formula for nonequidistant ordinates:

$$U_x = U_a \frac{(x-b)(x-c)\cdots(x-n)}{(a-b)(a-c)\cdots(a-n)} + U_b \frac{(x-a)(x-c)\cdots(x-n)}{(b-a)(b-c)\cdots(b-n)} + \cdots + U_n \frac{(x-a)(x-b)\cdots(x-m)}{(n-a)(n-b)\cdots(n-m)}$$

Newton's interpolation formula for equidistant ordinates:

$$U_{a+k} = U_a + k\Delta U_a + \frac{k(k-h)}{2!}\Delta^2 U_a + \frac{k(k-h)(k-2h)}{3!}\Delta^3 U_a + \cdots$$

Simple interpolation: For interpolation in the tables of this book, sufficient accuracy may be obtained by using Newton's formula to first differences. Then

$$U_{a+k} = U_a + k\frac{U_{a+h} - U_a}{h} \qquad k = h\frac{U_{a+k} - U_a}{U_{a+h} - U_a}$$

13. PRIME NUMBERS

A prime number is a number greater than 1 which is not exactly divisible by any number except itself and unity. The number of primes is infinite.

If p is a prime number and N is prime to p, then $N^{p-1} - 1$ is a multiple of p. (Fermat's Theorem.)

If p is a prime number, then $1 + (p-1)!$ is divisible by p. (Wilson's Theorem.)

SHORT TABLE OF PRIME NUMBERS

2	53	127	199	283	383	467	577	661	769	877	983
3	59	131	211	293	389	479	587	673	773	881	991
5	61	137	223	307	397	487	593	677	787	883	997
7	67	139	227	311	401	491	599	683	797	887	1009
11	71	149	229	313	409	499	601	691	809	907	1013
13	73	151	233	317	419	503	607	701	811	911	1019
17	79	157	239	331	421	509	613	709	821	919	1021
19	83	163	241	337	431	521	617	719	823	929	1031
23	89	167	251	347	433	523	619	727	827	937	1033
29	97	173	257	349	439	541	631	733	829	941	1039
31	101	179	263	353	443	547	641	739	839	947	1049
37	103	181	269	359	449	557	643	743	853	953	1051
41	107	191	271	367	457	563	647	751	857	967	1061
43	109	193	277	373	461	569	653	757	859	971	1063
47	113	197	281	379	463	571	659	761	863	977	1069

6 Formulas from Geometry

In the following, K = area, r = radius of the inscribed circle, R = radius of the circumscribed circle, S = lateral area, T = total area, and V = volume.

1. RIGHT TRIANGLE

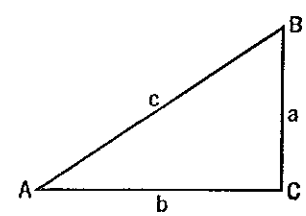
$A + B = C = 90°$

$A + B + C = 180°$

$c^2 = a^2 + b^2$

$a = \sqrt{(c+b)(c-b)}$

$K = \frac{1}{2}ab \qquad r = \frac{ab}{a+b+c} \qquad R = \frac{1}{2}c$



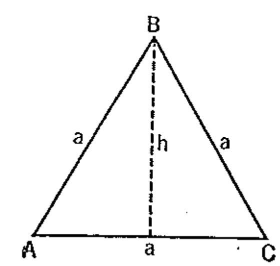
2. EQUILATERAL TRIANGLE

$A = B = C = 60°$

$A + B + C = 180°$

$h = \frac{\sqrt{3}}{2}a \qquad K = \frac{\sqrt{3}}{4}a^2$

$r = \frac{a}{2\sqrt{3}} \qquad R = \frac{a}{\sqrt{3}}$



3. OBLIQUE TRIANGLE

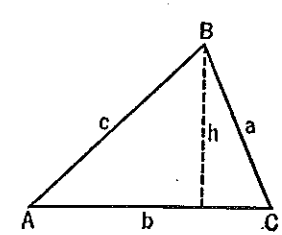
$A + B + C = 180°$

$h = c \sin A = a \sin C$

$K = \frac{1}{2}bh = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A$

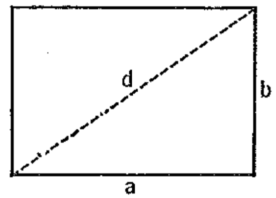
$K = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{1}{2}(a+b+c)$

$r = \frac{K}{s} \qquad R = \frac{abc}{4K}$



4. RECTANGLE

$$d = \sqrt{a^2 + b^2} \qquad K = ab$$

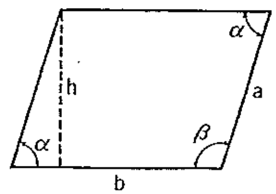


5. PARALLELOGRAM

$$\alpha + \beta = 180°$$

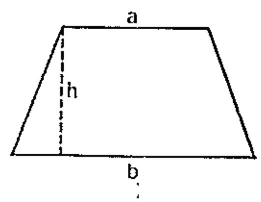
$$h = a \sin \alpha = a \sin \beta$$

$$K = bh = ab \sin \alpha = ab \sin \beta$$



6. TRAPEZOID

$$K = \tfrac{1}{2}(a + b)h$$



7. REGULAR POLYGON OF n SIDES

Let a = length of each side, θ = one of the equal angles.

$$\theta = \left(\frac{n-2}{n}\right) 180° \qquad a = 2r \tan \frac{180°}{n} = 2R \sin \frac{180°}{n}$$

$$r = \frac{a}{2} \operatorname{ctn} \frac{180°}{n} \qquad R = \frac{a}{2} \csc \frac{180°}{n}$$

$$K = \tfrac{1}{4} n a^2 \operatorname{ctn} \frac{180°}{n} = nr^2 \tan \frac{180°}{n} = \tfrac{1}{2} n R^2 \sin \frac{360°}{n}.$$

8. CIRCLE

Let R = radius, D = diameter, C = circumference.

$$C = 2\pi R = \pi D \qquad (\pi = 3.14159 \cdots)$$

$$K = \pi R^2 = \tfrac{1}{4}\pi D^2 = 0.7854 D^2$$

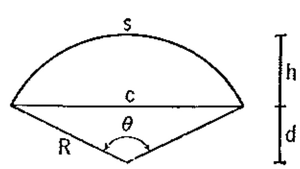
$$C = 2\sqrt{\pi K} = \frac{2K}{R} \qquad K = \frac{C^2}{4\pi} = \tfrac{1}{2} CR$$

9. SECTOR AND SEGMENT OF A CIRCLE

Let θ = central angle in radians ($\theta < \pi$), and s = length of arc subtended by θ.

$$h = R - d \qquad d = R - h$$

$$s = R\theta$$



$$d = R \cos \tfrac{1}{2}\theta = \tfrac{1}{2} c \operatorname{ctn} \tfrac{1}{2}\theta$$

$$= \tfrac{1}{2}\sqrt{4R^2 - c^2}$$

$$c = 2R \sin \tfrac{1}{2}\theta = 2d \tan \tfrac{1}{2}\theta = 2\sqrt{R^2 - d^2} = \sqrt{4h(2R - h)}$$

$$\theta = \frac{s}{R} = 2 \operatorname{Arccos} \frac{d}{R} = 2 \operatorname{Arctan} \frac{c}{2d} = 2 \operatorname{Arcsin} \frac{c}{2R}$$

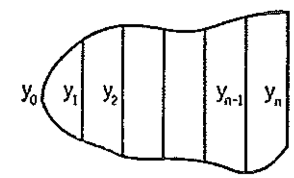
$$K(\text{sector}) = \tfrac{1}{2} Rs = \tfrac{1}{2} R^2 \theta$$

$$K(\text{segment}) = \tfrac{1}{2} R^2 (\theta - \sin \theta) = \tfrac{1}{2}(Rs - cd)$$

$$= R^2 \operatorname{Arccos} \frac{d}{R} - d\sqrt{R^2 - d^2}$$

10. AREA OF PLANE FIGURE BY APPROXIMATION

Divide the plane area into n strips by the equidistant parallel chords y_0, y_1, $y_2, \cdots, y_n$ (y_0 and y_n may be zero), and let h denote the common distance between them. Then, approximately,



Trapezoidal Rule:

$$K = h(\tfrac{1}{2}y_0 + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \tfrac{1}{2}y_n).$$

Also, if n is even,

Simpson's Rule:

$$K = \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n).$$

11. CUBE

Let a = length of each side.

$$T = 6a^2 \qquad \text{Diagonal of face} = a\sqrt{2}$$

$$V = a^3 \qquad \text{Diagonal of cube} = a\sqrt{3}$$

12. RECTANGULAR PARALLELEPIPED

Let a, b, c be the lengths of the sides.

$$T = 2(ab + ac + bc) \qquad V = abc$$

$$\text{Diagonal} = \sqrt{a^2 + b^2 + c^2}$$

13. PRISM OR CYLINDER

$$S = (\text{perimeter of right section}) \times (\text{lateral edge})$$

$$V = (\text{area of right section}) \times (\text{lateral edge})$$

$$= (\text{area of base}) \times (\text{altitude})$$

14. RIGHT CIRCULAR CYLINDER

Let R = radius of base and h = altitude.

$$S = 2\pi Rh \qquad V = \pi R^2h$$

15. PYRAMID OR CONE

$$V = \tfrac{1}{3}(\text{area of base}) \times (\text{altitude})$$

$$S(\text{regular pyramid}) = \tfrac{1}{2}(\text{perimeter of base}) \times (\text{slant height})$$

16. RIGHT CIRCULAR CONE

Let R = radius of base and h = altitude.

$$S = \pi R\sqrt{R^2 + h^2} \qquad V = \tfrac{1}{3}\pi R^2h$$

17. FRUSTUM OF REGULAR PYRAMID OR CONE

Let b, B = areas of the bases and h = altitude.

$$S = \tfrac{1}{2}(\text{sum of perimeters of bases}) \times (\text{slant height})$$

$$V = \tfrac{1}{3}h(b + B + \sqrt{bB})$$

18. PRISMATOID

Let b, B = areas of bases, M = area of midsection, h = altitude.

$$V = \tfrac{1}{6}h(b + B + 4M)$$

19. REGULAR POLYHEDRA OF EDGE *a*

Name	*Nature of Surface*	*Total Area*	*Volume*
Tetrahedron	4 equilateral triangles	$1.73205a^2$	$0.11785a^3$
Hexahedron (cube)	6 squares	$6.00000a^2$	$1.00000a^3$
Octahedron	8 equilateral triangles	$3.46410a^2$	$0.47140a^3$
Dodecahedron	12 pentagons	$20.64573a^2$	$7.66312a^3$
Icosahedron	20 equilateral triangles	$8.66025a^2$	$2.18170a^3$

20. SPHERE

Let R = radius, D = diameter, and h = altitude of segment or zone.

$$S = 4\pi R^2 = \pi D^2 \qquad V = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3$$

$$S(\text{zone}) = 2\pi Rh = \pi Dh$$

$$V(\text{spherical sector}) = \tfrac{2}{3}\pi R^2h = \tfrac{1}{6}\pi D^2h$$

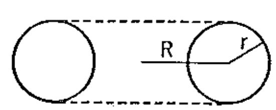
$$V(\text{spherical segment, one base of radius } a) = \tfrac{1}{3}\pi h^2(3R - h) = \tfrac{1}{6}\pi h(3a^2 + h^2)$$

$$V(\text{spherical segment, two bases of radii } a \text{ and } b) = \tfrac{1}{6}\pi h(3a^2 + 3b^2 + h^2)$$

21. CIRCULAR TORUS

$$S = 4\pi^2Rr$$

$$V = 2\pi^2Rr^2$$



22. THEOREMS OF PAPPUS

1. The area of a surface generated by revolving a plane curve about an external axis in its plane is equal to the product of the length of the curve and the length of the circumference of the circle described by its center of gravity.

2. The volume of a solid generated by revolving a plane area about an external axis in its plane is equal to the product of the area and the length of the circumference of the circle described by its center of gravity.

23. CENTROIDS

If a body has a center of symmetry, that point is the centroid.

If a body has an axis of symmetry, the centroid lies on that axis.

Body	*Location of Centroid*
Triangle	Intersection of medians
Rectangle	Intersection of diagonals
Quadrilateral	Trisect each side and construct a parallelogram with sides passing through pairs of adjacent third-points. The intersection of its diagonals is the desired centroid.
Area of semicircle of radius r	Distance from diameter $= \dfrac{4r}{3\pi}$
Arc of semicircle of radius r	Distance from diameter $= \dfrac{2r}{\pi}$
Area of semiellipse of altitude h	Distance from base $= \dfrac{4h}{3\pi}$
Area of parabolic segment of altitude h	Distance from base $= \frac{2}{5}h$
Volume of right circular cone	Distance from base $= \frac{1}{4}h$
Volume of hemisphere of radius r	Distance from center of sphere $= \frac{3}{8}r$
Volume of paraboloid of altitude h	Distance from base $= \frac{1}{3}h$

7 Formulas from Trigonometry

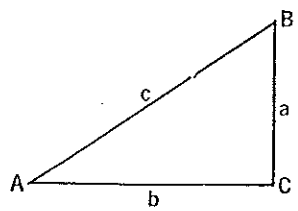
1. THE TRIGONOMETRIC FUNCTIONS

In the right triangle ABC,

$$\sin A = \frac{a}{c} \qquad \csc A = \frac{c}{a}$$

$$\cos A = \frac{b}{c} \qquad \sec A = \frac{c}{b}$$

$$\tan A = \frac{a}{b} \qquad \text{ctn}\, A = \frac{b}{a}$$



$$\text{exsec}\, A = \sec A - 1 \qquad \text{vers}\, A = 1 - \cos A$$

$$\text{covers}\, A = 1 - \sin A \qquad \text{hav}\, A = \tfrac{1}{2}\,\text{vers}\, A$$

2. THE SIGNS OF THE FUNCTIONS

Quadrant	*sin*	*cos*	*tan*	*ctn*	*sec*	*csc*
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

3. FUNCTIONS OF SPECIAL ANGLES

	sin	*cos*	*tan*	*ctn*	*sec*	*csc*
0°	0	1	0	....	1	....
30°	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
90°	1	0	....	0	....	1
120°	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	-2	$2/\sqrt{3}$
135°	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$1/2$	$-\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	$-2/\sqrt{3}$	2
180°	0	-1	0	....	-1	....
270°	-1	0	....	0	....	-1

4. REDUCTION FORMULAS

$$\sin\theta = +\cos(\theta - 90°) = -\sin(\theta - 180°) = -\cos(\theta - 270°)$$

$$\cos\theta = -\sin(\theta - 90°) = -\cos(\theta - 180°) = +\sin(\theta - 270°)$$

$$\tan\theta = -\text{ctn}(\theta - 90°) = +\tan(\theta - 180°) = -\text{ctn}(\theta - 270°)$$

$$\text{ctn}\,\theta = -\tan(\theta - 90°) = +\text{ctn}(\theta - 180°) = -\tan(\theta - 270°)$$

$$\sec\theta = -\csc(\theta - 90°) = -\sec(\theta - 180°) = +\csc(\theta - 270°)$$

$$\csc\theta = +\sec(\theta - 90°) = -\csc(\theta - 180°) = -\sec(\theta - 270°)$$

5. REDUCTION FORMULAS (continued)

	sin	*cos*	*tan*	*ctn*	*sec*	*csc*
$-\theta$	$-\sin\theta$	$+\cos\theta$	$-\tan\theta$	$-\text{ctn}\,\theta$	$+\sec\theta$	$-\csc\theta$
$90° + \theta$	$+\cos\theta$	$-\sin\theta$	$-\text{ctn}\,\theta$	$-\tan\theta$	$-\csc\theta$	$+\sec\theta$
$90° - \theta$	$+\cos\theta$	$+\sin\theta$	$+\text{ctn}\,\theta$	$+\tan\theta$	$+\csc\theta$	$+\sec\theta$
$180° + \theta$	$-\sin\theta$	$-\cos\theta$	$+\tan\theta$	$+\text{ctn}\,\theta$	$-\sec\theta$	$-\csc\theta$
$180° - \theta$	$+\sin\theta$	$-\cos\theta$	$-\tan\theta$	$-\text{ctn}\,\theta$	$-\sec\theta$	$+\csc\theta$
$270° + \theta$	$-\cos\theta$	$+\sin\theta$	$-\text{ctn}\,\theta$	$-\tan\theta$	$+\csc\theta$	$-\sec\theta$
$270° - \theta$	$-\cos\theta$	$-\sin\theta$	$+\text{ctn}\,\theta$	$+\tan\theta$	$-\csc\theta$	$-\sec\theta$
$360° + \theta$	$+\sin\theta$	$+\cos\theta$	$+\tan\theta$	$+\text{ctn}\,\theta$	$+\sec\theta$	$+\csc\theta$
$360° - \theta$	$-\sin\theta$	$+\cos\theta$	$-\tan\theta$	$-\text{ctn}\,\theta$	$+\sec\theta$	$-\csc\theta$

6. FUNDAMENTAL IDENTITIES

(Where a double sign occurs, the choice depends upon the quadrant in which the angle is located.)

$$\sin\theta\csc\theta = 1 \qquad \cos\theta\sec\theta = 1 \qquad \tan\theta\,\text{ctn}\,\theta = 1$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \text{ctn}^2\theta = \csc^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{ctn}\,\theta = \frac{\cos\theta}{\sin\theta} \qquad \tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta) \qquad \cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta)$$

$$\sin^3\theta = \tfrac{1}{4}(3\sin\theta - \sin 3\theta) \qquad \cos^3\theta = \tfrac{1}{4}(3\cos\theta + \cos 3\theta)$$

$$\sin^4\theta = \tfrac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta) \qquad \cos^4\theta = \tfrac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad \text{ctn}\, 2\theta = \frac{\text{ctn}^2\theta - 1}{2\,\text{ctn}\,\theta}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \qquad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\operatorname{ctn}\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \operatorname{ctn}(\alpha+\beta) = \frac{\operatorname{ctn}\beta\operatorname{ctn}\alpha - 1}{\operatorname{ctn}\beta+\operatorname{ctn}\alpha}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \qquad \operatorname{ctn}(\alpha-\beta) = \frac{\operatorname{ctn}\beta\operatorname{ctn}\alpha + 1}{\operatorname{ctn}\beta-\operatorname{ctn}\alpha}$$

$$\sin\alpha+\sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$\sin\alpha-\sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

$$\cos\alpha+\cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$\cos\alpha-\cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$$

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$

$$\tan\alpha+\tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \qquad \tan\alpha-\tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$

$$\operatorname{ctn}\alpha+\operatorname{ctn}\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \qquad \operatorname{ctn}\alpha-\operatorname{ctn}\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$$

$$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)} \qquad \frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta} = \operatorname{ctn}\tfrac{1}{2}(\beta-\alpha)$$

$$\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} = \tan\tfrac{1}{2}(\alpha+\beta) \qquad \frac{\sin\alpha-\sin\beta}{\cos\alpha+\cos\beta} = \tan\tfrac{1}{2}(\alpha-\beta)$$

$$e^{ix} = \cos x + i\sin x \qquad (i = \sqrt{-1})$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\tan x = -i\left(\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}\right) = -i\left(\frac{e^{2ix}-1}{e^{2ix}+1}\right)$$

7. EQUIVALENT EXPRESSIONS FOR SIN θ, COS θ, AND TAN θ

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \frac{\tan\theta}{\pm\sqrt{1+\tan^2\theta}} = \frac{1}{\pm\sqrt{1+\operatorname{ctn}^2\theta}} = \frac{\pm\sqrt{\sec^2\theta-1}}{\sec\theta}$$

$$= \frac{1}{\csc\theta} = \cos\theta\tan\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}} = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} = \frac{1}{\pm\sqrt{1+\tan^2\theta}} = \frac{\operatorname{ctn}\theta}{\pm\sqrt{1+\operatorname{ctn}^2\theta}} = \frac{1}{\sec\theta}$$

$$= \frac{\pm\sqrt{\csc^2\theta-1}}{\csc\theta} = \sin\theta\operatorname{ctn}\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}} = \cos^2\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}\theta$$

$$\tan\theta = \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}} = \frac{\pm\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{1}{\operatorname{ctn}\theta} = \pm\sqrt{\sec^2\theta-1}$$

$$= \frac{1}{\pm\sqrt{\csc^2\theta-1}} = \frac{\sin\theta}{\cos\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$$

8. RADIAN MEASURE

A *radian* is an angle which, if its vertex is placed at the center of a circle, intercepts an arc equal in length to the radius.

$$180° = \pi \text{ radians} \qquad 1° = \frac{\pi}{180}\text{ radians} \qquad 1 \text{ radian} = \frac{180°}{\pi}$$

Approximately,

$1° = 0.01745\ 32925$ radian

$1' = 0.00029\ 08882$ radian

$1'' = 0.00000\ 48481$ radian

1 radian $= 57.29577\ 95131°$

$= 57°\ 17'\ 44.80625''$

Degrees	Radians	Degrees	Radians	Degrees	Radians
30	$\pi/6$	135	$3\pi/4$	270	$3\pi/2$
45	$\pi/4$	150	$5\pi/6$	300	$5\pi/3$
60	$\pi/3$	210	$7\pi/6$	315	$7\pi/4$
90	$\pi/2$	225	$5\pi/4$	330	$11\pi/6$
120	$2\pi/3$	240	$4\pi/3$	360	2π

9. THE MIL SYSTEM OF ANGULAR MEASURE

1600 mils = 90°   1000 mils = 56.25°

1 mil = 0.05625° = 3.375′ = 202.5″

1° = 17.777778 mils   1 mil = 0.00098175 radian

1′ = 0.296296 mil   1 radian = 1018.6 mils

The length of the arc or chord subtended on the circumference of a circle of radius r by a small central angle θ measured in mils is, approximately,

$$S = \frac{r\theta}{1000}.$$

10. PLANE TRIANGLE FORMULAS

In the following, A, B, C represent the angles of any plane triangle, a, b, c the corresponding opposite sides, and $s = \frac{1}{2}(a + b + c)$.

Radius of the inscribed circle:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Radius of the circumscribed circle:

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

Law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Law of tangents:

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \qquad \frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)}$$

Half-angle formulas:

$$\tan\tfrac{1}{2}A = \frac{r}{s-a} \qquad \tan\tfrac{1}{2}B = \frac{r}{s-b} \qquad \tan\tfrac{1}{2}C = \frac{r}{s-c}$$

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\sin\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}} \qquad \cos\tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}$$

$$\sin\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}} \qquad \cos\tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}$$

Area:

$$K = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}bc\sin A$$

$$K = \frac{a^2\sin B\sin C}{2\sin A} = \frac{b^2\sin C\sin A}{2\sin B} = \frac{c^2\sin A\sin B}{2\sin C}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)} = rs \qquad K = \frac{abc}{4R}$$

Newton's formulas:

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} \qquad \frac{a+c}{b} = \frac{\cos\frac{1}{2}(A-C)}{\sin\frac{1}{2}B}$$

$$\frac{b+c}{a} = \frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}$$

Mollweide's formulas:

$$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C} \qquad \frac{a-c}{b} = \frac{\sin\frac{1}{2}(A-C)}{\cos\frac{1}{2}B}$$

$$\frac{b-c}{a} = \frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A}$$

11. SUMMARY OF THE SOLUTION OF OBLIQUE TRIANGLES

Given	*Nonlogarithmic Solution*	*Logarithmic Solution*	*Check*	*Area*
Two angles and one side	Law of sines	Law of sines	Newton's formula, or law of tangents	$\dfrac{a^2 \sin B \sin C}{2 \sin A}$
Two sides and their included angle	Law of cosines	Law of tangents, then law of sines	Newton's formula, or law of sines	$\dfrac{ab \sin C}{2}$
Three sides	Law of cosines	Tangents of half angles	$A + B + C = 180°$	$\sqrt{s(s-a)(s-b)(s-c)}$
Two sides and an opposite angle	Law of sines	Law of sines	Newton's formula, or law of tangents	$\dfrac{ab \sin C}{2}$

12. INVERSE TRIGONOMETRIC FUNCTIONS

Principal values:

$$-\frac{\pi}{2} \leqq \text{Arcsin } x \leqq \frac{\pi}{2}, \qquad -1 \leqq x \leqq 1$$

$$-\frac{\pi}{2} < \text{Arctan } x < \frac{\pi}{2}, \qquad -\infty < x < \infty$$

$$0 \leqq \text{Arccos } x \leqq \pi, \qquad -1 \leqq x \leqq 1$$

$$0 < \text{Arcctn } x < \pi, \qquad -\infty < x < \infty$$

$$0 \leqq \text{Arcsec } x < \frac{\pi}{2}, \qquad x \geqq 1$$

$$-\pi \leqq \text{Arcsec } x < -\frac{\pi}{2}, \qquad x \leqq -1$$

$$0 < \text{Arccsc } x \leqq \frac{\pi}{2}, \qquad x \geqq 1$$

$$-\pi < \text{Arccsc } x \leqq -\frac{\pi}{2}, \qquad x \leqq -1$$

Note: There is disagreement on the definitions of the principal values of arcctn x, arcsec x, and arccsc x for negative values of x.

Fundamental identities involving principal values:

$$\text{Arcsin } x + \text{Arccos } x = \frac{\pi}{2}$$

$$\text{Arctan } x + \text{Arcctn } x = \frac{\pi}{2}$$

If $\theta =$ Arcsin x, then

$$\sin\theta = x \qquad \tan\theta = \frac{x}{\sqrt{1-x^2}} \qquad \sec\theta = \frac{1}{\sqrt{1-x^2}}$$

$$\cos\theta = \sqrt{1-x^2} \qquad \text{ctn}\,\theta = \frac{\sqrt{1-x^2}}{x} \qquad \csc\theta = \frac{1}{x}$$

If $\theta =$ Arccos x, then

$$\sin\theta = \sqrt{1-x^2} \qquad \tan\theta = \frac{\sqrt{1-x^2}}{x} \qquad \sec\theta = \frac{1}{x}$$

$$\cos\theta = x \qquad \text{ctn}\,\theta = \frac{x}{\sqrt{1-x^2}} \qquad \csc\theta = \frac{1}{\sqrt{1-x^2}}$$

If $\theta =$ Arctan x, then

$$\sin\theta = \frac{x}{\sqrt{1+x^2}} \qquad \tan\theta = x \qquad \sec\theta = \sqrt{1+x^2}$$

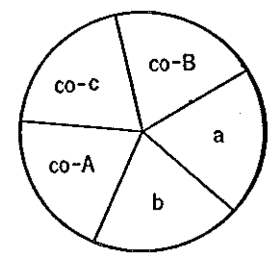
$$\cos\theta = \frac{1}{\sqrt{1+x^2}} \qquad \text{ctn}\,\theta = \frac{1}{x} \qquad \csc\theta = \frac{\sqrt{1+x^2}}{x}$$

13. SPHERICAL TRIANGLE FORMULAS

In the following, A, B, C represent the angles of any spherical triangle, a, b, c the corresponding opposite sides, $S = \frac{1}{2}(A + B + C)$, and $s = \frac{1}{2}(a + b + c)$.

Napier's rules, $C = 90°$:

1. The sine of any middle part is equal to the product of the *tangents* of the *adjacent* parts.
2. The sine of any middle part is equal to the product of the *cosines* of the *opposite* parts.



Law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Law of cosines for sides:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Law of cosines for angles:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

Law of tangents:

$$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}(a+b)} \qquad \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)} = \frac{\tan \frac{1}{2}(a-c)}{\tan \frac{1}{2}(a+c)}$$

$$\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{\tan \frac{1}{2}(b-c)}{\tan \frac{1}{2}(b+c)}$$

Half-angle formulas:

$$\tan \tfrac{1}{2}A = \frac{k}{\sin (s-a)} \qquad \tan \tfrac{1}{2}B = \frac{k}{\sin (s-b)} \qquad \tan \tfrac{1}{2}C = \frac{k}{\sin (s-c)}$$

where $$k^2 = \frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}$$

Half-side formulas:

$$\tan \tfrac{1}{2}a = K \cos (S-A) \qquad \tan \tfrac{1}{2}b = K \cos (S-B)$$

$$\tan \tfrac{1}{2}c = K \cos (S-C)$$

where $$K^2 = \frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}$$

Napier's analogies:

$$\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c} \qquad \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A-B)}{\operatorname{ctn} \frac{1}{2}C}$$

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c} \qquad \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A+B)}{\operatorname{ctn} \frac{1}{2}C}$$

Gauss's formulas:

$$\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C} \qquad \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}$$

$$\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}C} \qquad \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C}$$

The law of species:

1. If $A > B > C$, then $a > b > c$.
2. Half the sum of any two sides and half the sum of the opposite angles are of the same species.
3. A side (angle) which differs from 90° more than another side (angle) does is of the same species as its opposite angle (side).

14. SUMMARY OF THE SOLUTION OF SPHERICAL TRIANGLES

Given	*Solution*	*Check*
Three sides	Half-angle formulas	Law of sines
Three angles	Half-side formulas	Law of sines
Two sides and their included angle	Napier's analogies (find sum and difference of unknown angles), then law of sines	Gauss's formula
Two angles and their included side	Napier's analogies (find sum and difference of unknown sides), then law of sines	Gauss's formula
Two sides and an opposite angle	Law of sines, then Napier's analogies	Gauss's formula
Two angles and an opposite side	Law of sines, then Napier's analogies	Gauss's formula

15. HAVERSINES

$$\operatorname{hav} \theta = \tfrac{1}{2} \operatorname{vers} \theta = \tfrac{1}{2}(1 - \cos \theta) = \sin^2 \tfrac{1}{2}\theta$$

$$\operatorname{hav}(-\theta) = \operatorname{hav} \theta$$

$$\operatorname{hav} (180° - \theta) = \operatorname{hav} (180° + \theta) = 1 - \operatorname{hav} \theta$$

Let A, B, C be the angles of a spherical triangle, a, b, c the corresponding opposite sides, and $s = \frac{1}{2}(a + b + c)$. Then

$$\operatorname{hav} a = \operatorname{hav} (b - c) + \sin b \sin c \operatorname{hav} A$$

$$\operatorname{hav} A = \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}$$

$$= \frac{\operatorname{hav} a - \operatorname{hav} (b-c)}{\sin b \sin c}$$

$$= \operatorname{hav} [180° - (B + C)] + \sin B \sin C \operatorname{hav} a$$

8 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{ctnh} x = \frac{1}{\tanh x}$$

$$\sinh(-x) = -\sinh x \qquad \operatorname{ctnh}(-x) = -\operatorname{ctnh} x$$

$$\cosh(-x) = \cosh x \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$

$$\tanh(-x) = -\tanh x \qquad \operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{ctnh} x = \frac{\cosh x}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1)$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1 \qquad \cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1)$$

$$\operatorname{ctnh}^2 x - \operatorname{csch}^2 x = 1 \qquad \operatorname{csch}^2 x - \operatorname{sech}^2 x = \operatorname{csch}^2 x \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \qquad \tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

$$\sinh(x+y) + \sinh(x-y) = 2 \sinh x \cosh y$$

$$\sinh(x+y) - \sinh(x-y) = 2 \cosh x \sinh y$$

$$\cosh(x+y) + \cosh(x-y) = 2 \cosh x \cosh y$$

$$\cosh(x+y) - \cosh(x-y) = 2 \sinh x \sinh y$$

$$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y = \cosh^2 x - \cosh^2 y$$

$$\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y = \cosh^2 x + \sinh^2 y$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x+y) \cosh \tfrac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2 \cosh \tfrac{1}{2}(x+y) \sinh \tfrac{1}{2}(x-y)$$

$$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x+y) \cosh \tfrac{1}{2}(x-y)$$

$$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x+y) \sinh \tfrac{1}{2}(x-y)$$

$$\sinh 2x = 2 \sinh x \cosh x \qquad \sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \qquad \cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \qquad \tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

$$\sinh \tfrac{1}{2}x = \pm\sqrt{\tfrac{1}{2}(\cosh x - 1)} \qquad \cosh \tfrac{1}{2}x = \sqrt{\tfrac{1}{2}(\cosh x + 1)}$$

$$\tanh \tfrac{1}{2}x = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right) \qquad \operatorname{ctnh}^{-1} x = \tfrac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) \quad (x^2 > 1)$$

$$\cosh^{-1} x = \log_e\left(x \pm \sqrt{x^2-1}\right) \quad (x \geqq 1) \qquad \operatorname{sech}^{-1} x = \log_e\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right) \quad (0 < x \leqq 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \quad (x^2 < 1) \qquad \operatorname{csch}^{-1} x = \log_e\left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right)$$

$$\sinh ix = i \sin x \qquad \sinh x = -i \sin ix$$

$$\cosh ix = \cos x \qquad \cosh x = \cos ix$$

$$\tanh ix = i \tan x \qquad \tanh x = -i \tan ix$$

$$\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$$

$$\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$$

$$\sinh(x-iy) = \sinh x \cos y - i \cosh x \sin y$$

$$\cosh(x-iy) = \cosh x \cos y - i \sinh x \sin y$$

$$\sinh(x + \tfrac{1}{2}\pi i) = i \cosh x \qquad \cosh(x + \tfrac{1}{2}\pi i) = i \sinh x$$

$$\sinh(x + \pi i) = -\sinh x \qquad \cosh(x + \pi i) = -\cosh x$$

$$\sinh(x + 2\pi i) = \sinh x \qquad \cosh(x + 2\pi i) = \cosh x$$

9 Formulas from Plane Analytic Geometry

In the following formulas P_1 has the coordinates (x_1,y_1) and P_2 has the coordinates (x_2,y_2).

1. POINTS AND SLOPES

Length of segment P_1P_2: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Point dividing P_1P_2 in ratio $\frac{r}{s}$: $x = \dfrac{rx_2 + sx_1}{r+s}, \quad y = \dfrac{ry_2 + sy_1}{r+s}$

Midpoint of P_1P_2: $x = \frac{1}{2}(x_1 + x_2), \quad y = \frac{1}{2}(y_1 + y_2)$

Slope of P_1P_2: $m = \dfrac{y_1 - y_2}{x_1 - x_2}$

Angle between two lines: $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1m_2}$

For parallel lines, $m_1 = m_2$.

For perpendicular lines, $m_1m_2 = -1$.

Area of triangle $P_1P_2P_3 = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)$.

2. STRAIGHT LINES

Line parallel to the y-axis: $x = h$

Line parallel to the x-axis: $y = k$

Point-slope form: $y - y_1 = m(x - x_1)$

Slope-intercept form: $y = mx + b$

Two-point form: $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Intercept form: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Normal form: $x\cos\omega + y\sin\omega = p$

General form: $Ax + By + C = 0$

To reduce $Ax + By + C = 0$ to the normal form, divide the members of the equation by $\pm\sqrt{A^2 + B^2}$. Choose the sign of the radical opposite to that of C. If $C = 0$, choose the sign to be the same as the sign of B.

Distance from $Ax + By + C = 0$ to P_1: $d = \dfrac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$

3. CIRCLES

Center at origin, radius r: $x^2 + y^2 = r^2$

Center at (h,k), radius r: $(x - h)^2 + (y - k)^2 = r^2$

General form: $Ax^2 + Ay^2 + Dx + Ey + F = 0$

4. PARABOLAS ($e = 1$)

Distance from the vertex to the focus $= p$, length of the latus rectum $= 4p$.

Vertex at origin, focus at $(p,0)$: $y^2 = 4px$

Vertex at origin, focus at $(0,p)$: $x^2 = 4py$

Vertex at (h,k), focus at $(h + p,k)$: $(y - k)^2 = 4p(x - h)$

Vertex at (h,k), focus at $(h,k + p)$: $(x - h)^2 = 4p(y - k)$

General form, axis parallel to x-axis: $Cy^2 + Dx + Ey + F = 0$

General form, axis parallel to y-axis: $Ax^2 + Dx + Ey + F = 0$ or $y = ax^2 + bx + c$

General form, axis oblique to x- and y-axes:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (B^2 - 4AC = 0)$$

5. ELLIPSES ($e < 1$)

Length of major axis $= 2a$, length of minor axis $= 2b$, length of latus rectum $= \dfrac{2b^2}{a}$, distance from center to either focus $= \sqrt{a^2 - b^2}$, eccentricity $= \dfrac{\sqrt{a^2 - b^2}}{a}$.

Center at origin, foci on x-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Center at origin, foci on y-axis: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

Center at (h,k), major axis parallel to x-axis: $$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Center at (h,k), major axis parallel to y-axis: $$\frac{(x-h^2)}{b^2}+\frac{(y-k)^2}{a^2}=1$$

General form, axes parallel to x- and y-axes:
$$Ax^2+Cy^2+Dx+Ey+F=0 \qquad (AC>0)$$

General form, axes oblique to x- and y-axes:
$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \qquad (B^2-4AC<0)$$

6. HYPERBOLAS ($e>1$)

Length of transverse axis $=2a$, length of conjugate axis $=2b$, length of latus rectum $=\frac{2b^2}{a}$, distance from center to either focus $=\sqrt{a^2+b^2}$, eccentricity $=\frac{\sqrt{a^2+b^2}}{a}$.

Center at origin, foci on x-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Center at origin, foci on y-axis: $$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Center at origin, x- and y-axes as asymptotes: $xy=c$

Center at (h,k), transverse axis parallel to x-axis: $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Center at (h,k), transverse axis parallel to y-axis: $$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

General form, axes parallel to x- and y-axes:
$$Ax^2+Cy^2+Dx+Ey+F=0 \qquad (AC<0)$$

General form, axes oblique to x- and y-axes:
$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \qquad (B^2-4AC>0)$$

7. TRANSFORMATION OF COORDINATES

Translation: $\begin{cases} x=x'+h \\ y=y'+k \end{cases}$ [Coordinates of the new origin in terms of the old coordinates $=(h,k)$.]

Rotation: $\begin{cases} x=x'\cos\theta-y'\sin\theta \\ y=x'\sin\theta+y'\cos\theta \end{cases}$ [The x'-axis makes an angle θ with the x-axis.]

To remove the xy-term from the equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, rotate the axes through the acute angle arctan m, where m is the positive root of
$$Bm^2+2(A-C)m-B=0.$$

8. TANGENTS

The slope m of the tangent line may be obtained by differentiation, or by substituting the coordinates of the point of tangency in the following equations of tangent lines.

To the parabola $y^2=4px$: $$y=mx+\frac{p}{m}$$

To the parabola $x^2=4py$: $$y=mx-pm^2$$

To the circle $x^2+y^2=r^2$: $$y=mx\pm r\sqrt{m^2+1}$$

To the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$: $$y=mx\pm\sqrt{a^2m^2+b^2}$$

To the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$: $$y=mx\pm\sqrt{a^2m^2-b^2}$$

To the hyperbola $xy=c$: $$y=mx\pm 2\sqrt{-cm}$$

To $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ at the point (x_1,y_1):
$$Ax_1x+B\left(\frac{x_1y+y_1x}{2}\right)+Cy_1y+D\left(\frac{x+x_1}{2}\right)+E\left(\frac{y+y_1}{2}\right)+F=0$$

9. POLAR COORDINATES

If (x,y) are the rectangular coordinates and (ρ,θ) the polar coordinates of a point, then

$$x=\rho\cos\theta \qquad \rho=\sqrt{x^2+y^2} \qquad \sin\theta=\frac{y}{\sqrt{x^2+y^2}}$$

$$y=\rho\sin\theta \qquad \theta=\arctan\frac{y}{x} \qquad \cos\theta=\frac{x}{\sqrt{x^2+y^2}}$$

Straight line, normal form: $\rho\cos(\theta-\omega)=p$

Circle, center at pole, radius a: $\rho=a$

Circle through pole, center at $(a,0)$: $\rho=2a\cos\theta$

Circle through pole, center at $\left(a,\frac{\pi}{2}\right)$: $\rho=2a\sin\theta$

Conic section, focus at pole, distance from directrix to focus = $2p$:

$$\rho = \frac{2ep}{1 - e\cos\theta} \qquad \text{(Directrix to left of pole)}$$

$$\rho = \frac{2ep}{1 + e\cos\theta} \qquad \text{(Directrix to right of pole)}$$

$$\rho = \frac{2ep}{1 - e\sin\theta} \qquad \text{(Directrix below pole)}$$

$$\rho = \frac{2ep}{1 + e\sin\theta} \qquad \text{(Directrix above pole)}$$

10. GENERAL EQUATION OF THE SECOND DEGREE

The character of the graph of any equation of the second degree,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

is indicated by the values of Δ and d, where

$$\Delta = B^2 - 4AC \quad \text{and} \quad d = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix}$$

Case 1. $\Delta = 0$, $d = 0$: Two parallel, coincident, or imaginary lines
Case 2. $\Delta \neq 0$, $d = 0$: Two intersecting lines or a point
Case 3. $\Delta = 0$, $d \neq 0$: Parabola
Case 4. $\Delta < 0$, $d \neq 0$: Ellipse, real or imaginary
Case 5. $\Delta > 0$, $d \neq 0$: Hyperbola

In Cases 4 and 5, the center $C(x_o, y_o)$ is given by the solution of the equations $2Ax + By + D = 0$, $Bx + 2Cy + E = 0$. The equations of the axes of the conic are

$$y - y_o = m(x - x_o) \qquad y - y_o = -\frac{1}{m}(x - x_o)$$

where m is the positive root of

$$Bm^2 + 2(A - C)m - B = 0.$$

10 Formulas from Solid Analytic Geometry

1. POINTS AND LINES

Let α, β, γ be the angles that P_1P_2 or any parallel line makes with the x-, y-, and z-axes, respectively.

Midpoint of P_1P_2: $x = \frac{1}{2}(x_1 + x_2)$, $y = \frac{1}{2}(y_1 + y_2)$, $z = \frac{1}{2}(z_1 + z_2)$

Length of P_1P_2: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Direction cosines of P_1P_2:

$$\cos\alpha = \frac{x_2 - x_1}{d}, \quad \cos\beta = \frac{y_2 - y_1}{d}, \quad \cos\gamma = \frac{z_2 - z_1}{d}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

If $a : b : c = \cos\alpha : \cos\beta : \cos\gamma$, then

$$\cos\alpha = \frac{a}{\pm\sqrt{a^2 + b^2 + c^2}}, \quad \cos\beta = \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}},$$

$$\cos\gamma = \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}$$

Angle between two lines with direction angles $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$:

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2$$

Two lines are parallel if $\alpha_1 = \alpha_2, \beta_1 = \beta_2, \gamma_1 = \gamma_2$

or $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Two lines are perpendicular if

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$$

or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

2. PLANES

General form: $Ax + By + Cz + D = 0$

Perpendicular to xy-plane: $Ax + By + D = 0$

Perpendicular to yz-plane: $By + Cz + D = 0$

Perpendicular to xz-plane: $Ax + Cz + D = 0$

Perpendicular to x-axis: $Ax + D = 0$

Perpendicular to y-axis: $By + D = 0$

Perpendicular to z-axis: $Cz + D = 0$

Intercept form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Normal form: $x \cos \alpha + y \cos \beta + z \cos \gamma = p$

Here α, β, γ are the direction angles of a normal to the plane and p is the perpendicular distance from the origin to the plane.

To reduce $Ax + By + Cz + D = 0$ to the normal form, divide the members of the equation by $\pm\sqrt{A^2 + B^2 + C^2}$, choosing the sign of the radical opposite to that of D. If $D = 0$, choose the sign the same as that of C, or the same as the sign of B if $C = D = 0$.

Angle between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$:

$$\cos \theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \pm\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

The planes are parallel if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

The planes are perpendicular if $A_1A_2 + B_1B_2 + C_1C_2 = 0$

Distance from plane to P_1: $d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$

3. STRAIGHT LINES

General form: $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

Parametric form, line through P_1:

$$x = x_1 + t \cos \alpha, \qquad y = y_1 + t \cos \beta, \qquad z = z_1 + t \cos \gamma$$

Symmetric form, line through P_1: $\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}$

Two-point form: $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

4. SPECIAL SURFACES

Sphere, center at origin, radius r: $x^2 + y^2 + z^2 = r^2$

Sphere, center at (h, k, l), radius r: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Cylinder: Any equation in which one variable is lacking is a cylinder with elements parallel to the axis of the missing variable.

Cone: Any equation which is homogeneous in the variables x, y, and z is a cone with vertex at the origin.

Quadric surfaces:

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Hyperboloid of two sheets: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Elliptic paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$

Hyperbolic paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$

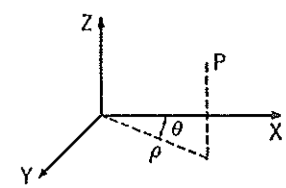
5. CYLINDRICAL COORDINATES

If (ρ, θ, z) are the cylindrical coordinates and (x, y, z) are the rectangular coordinates of a point P, then

$x = \rho \cos \theta \qquad \rho = \sqrt{x^2 + y^2}$

$y = \rho \sin \theta \qquad \theta = \arctan \frac{y}{x}$

$z = z$



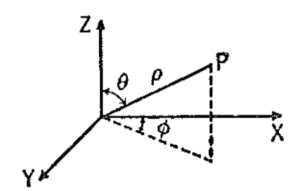
6. SPHERICAL COORDINATES

If (ρ, θ, ϕ) are the spherical coordinates and (x, y, z) are the rectangular coordinates of a point P, then

$x = \rho \sin \theta \cos \phi \qquad \phi = \arctan \frac{y}{x}$

$y = \rho \sin \theta \sin \phi \qquad \rho = \sqrt{x^2 + y^2 + z^2}$

$z = \rho \cos \theta \qquad \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}$



11 Curves for Reference

This table contains the graphs of certain standard functions studied in elementary mathematics. Graphs of many other functions may be obtained from these by means of suitable transformations. The more important transformations for plane curves are summarized in the following theorems.

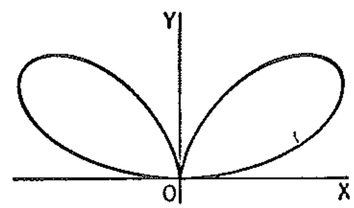
RECTANGULAR COORDINATES

1. If x is replaced by $-x$, the new curve is the reflection of the former in the y-axis.
2. If y is replaced by $-y$, the new curve is the reflection of the former in the x-axis.
3. If x is replaced by $-x$ and y is replaced by $-y$, the new curve is the reflection of the former in the origin.
4. If x and y are interchanged, the new curve is the reflection of the former in the line $y = x$.
5. If x is replaced by y and y is replaced by $-x$, the new curve is the former rotated about the origin through 90°.
6. If x is replaced by $-y$ and y is replaced by x, the new curve is the former rotated about the origin through $-90°$.
7. If x is replaced by $(x - h)$, the new curve is the former translated a distance h in the x-direction.
8. If y is replaced by $(y - k)$, the new curve is the former translated a distance k in the y-direction.
9. If x is replaced by $\frac{x}{a}$, all the abscissas are multiplied by a.
10. If y is replaced by $\frac{y}{b}$, all the ordinates are multiplied by b.

POLAR COORDINATES

1. If θ is replaced by $-\theta$, the new curve is the reflection of the former in the polar axis.
2. If θ is replaced by $(\pi + \theta)$, the new curve is the reflection of the former in the pole.

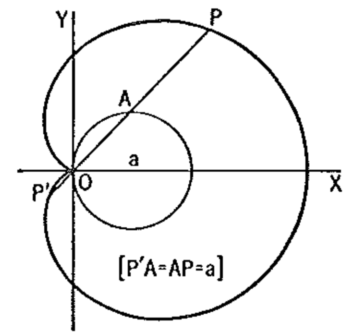
3. If θ is replaced by $(\pi - \theta)$, the new curve is the reflection of the former in the 90° axis.
4. If ρ is replaced by $-\rho$, the new curve is the reflection of the former in the pole.
5. If ρ is replaced by $-\rho$ and θ is replaced by $-\theta$, the new curve is the reflection of the former in the 90° axis.
6. If ρ is replaced by $-\rho$ and θ is replaced by $(\pi - \theta)$, the new curve is the reflection of the former in the polar axis.
7. If θ is replaced by $(\theta - \alpha)$, the new curve is the former rotated about the pole through the angle α.



BIFOLIUM

$$(x^2 + y^2)^2 = ax^2y$$

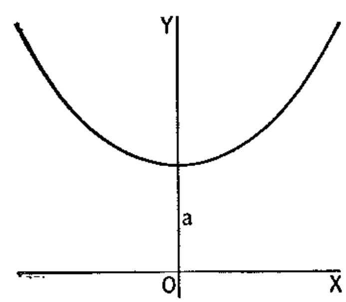
$$\rho = a \sin\theta \cos^2\theta$$



CARDIOID

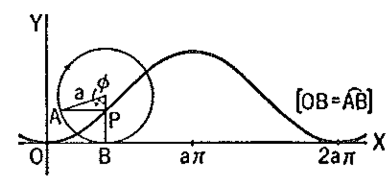
$$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$$

$$\rho = a(\cos\theta + 1) \text{ or } \rho = a(\cos\theta - 1)$$



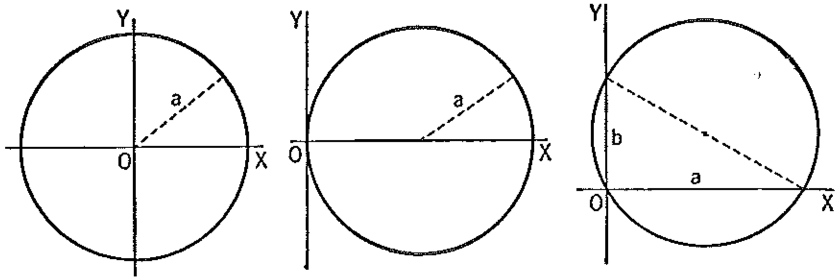
CATENARY

$$y = \frac{a}{2}(e^{x/a} + e^{-x/a}) = a \cosh\frac{x}{a}$$



COMPANION TO THE CYCLOID

$$x = a\varphi,\ y = a(1 - \cos\varphi)$$



CIRCLES

$x^2 + y^2 = a^2$

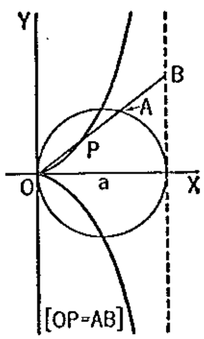
$\rho = a$

$x^2 + y^2 = 2ax$

$\rho = 2a\cos\theta$

$x^2 + y^2 = ax + by$

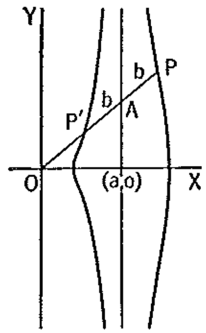
$\rho = a\cos\theta + b\sin\theta$



CISSOID OF DIOCLES

$y^2(a - x) = x^3$

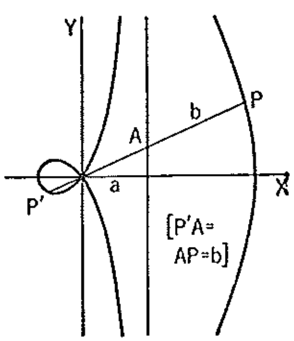
$\rho = a\sin\theta\tan\theta$



CONCHOID OF NICOMEDES ($a > b$)

$(x - a)^2(x^2 + y^2) = b^2x^2$

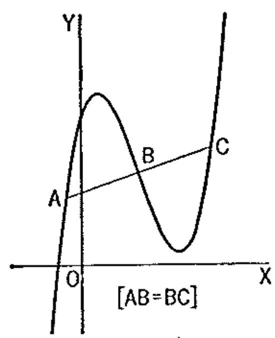
$\rho = a\sec\theta \pm b$



CONCHOID OF NICOMEDES ($a < b$)

$(x - a)^2(x^2 + y^2) = b^2x^2$

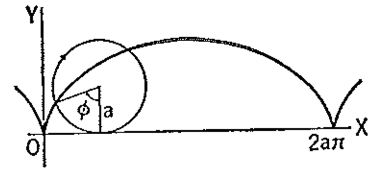
$\rho = a\sec\theta \pm b$



CUBICAL PARABOLA ($a > 0$)

$y = ax^3 + bx^2 + cx + d$

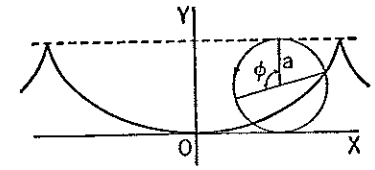
$\left[\text{abscissa of } B = -\dfrac{b}{3a}\right]$



CYCLOID, ORDINARY CASE

$x = a \operatorname{arcvers}\dfrac{y}{a} - \sqrt{2ay - y^2}$

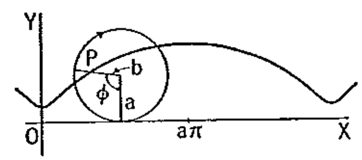
$x = a(\varphi - \sin\varphi), \quad y = a(1 - \cos\varphi)$



CYCLOID, VERTEX AT ORIGIN

$x = a \operatorname{arcvers}\dfrac{y}{a} + \sqrt{2ay - y^2}$

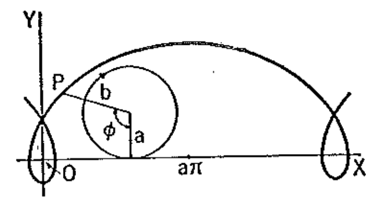
$x = a(\varphi + \sin\varphi), \quad y = a(1 - \cos\varphi)$



CYCLOID, CURTATE

$\begin{cases} x = a\varphi - b\sin\varphi \\ y = a - b\cos\varphi \end{cases}$

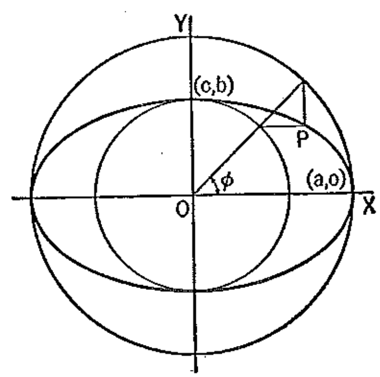
$a > b$



CYCLOID, PROLATE

$\begin{cases} x = a\varphi - b\sin\varphi \\ y = a - b\cos\varphi \end{cases}$

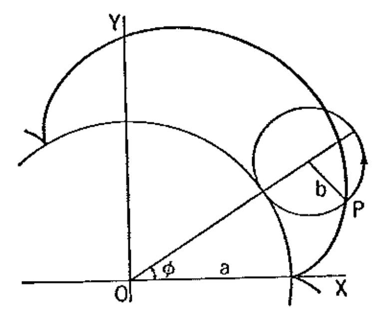
$a < b$



ELLIPSE

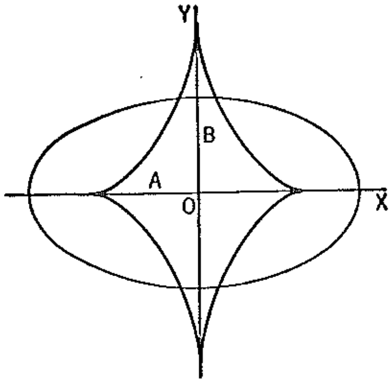
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$x = a\cos\varphi, \quad y = b\sin\varphi$



EPICYCLOID

$\begin{cases} x = (a+b)\cos\varphi - b\cos\dfrac{a+b}{b}\varphi \\ y = (a+b)\sin\varphi - b\sin\dfrac{a+b}{b}\varphi \end{cases}$

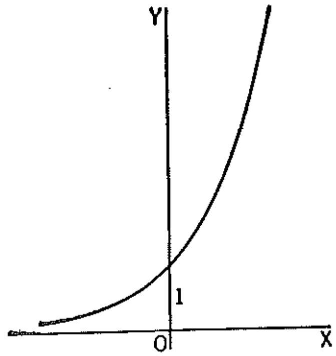


EVOLUTE OF ELLIPSE

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

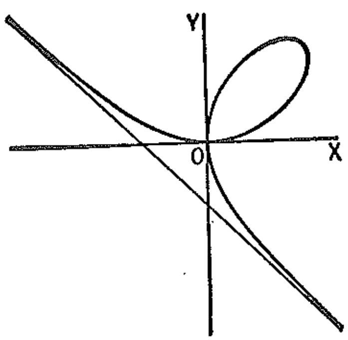
$$x = A \cos^3 \theta, \quad y = B \sin^3 \theta$$

$$\left[A = \frac{(a^2 - b^2)}{a}, \; B = \frac{(a^2 - b^2)}{b} \right]$$



EXPONENTIAL CURVE ($a > 0$)

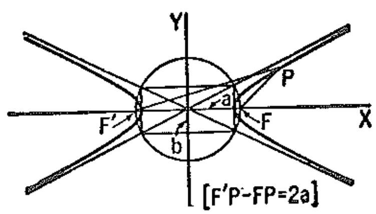
$$y = e^{ax}$$



FOLIUM OF DESCARTES

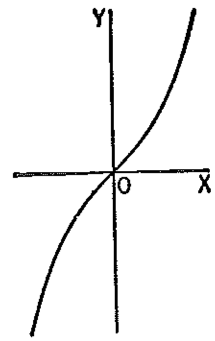
$$x^3 + y^3 = 3axy$$

[asymptote: $x + y + a = 0$]



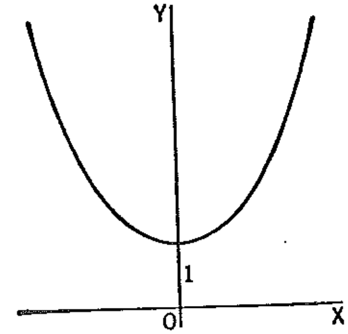
HYPERBOLA

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



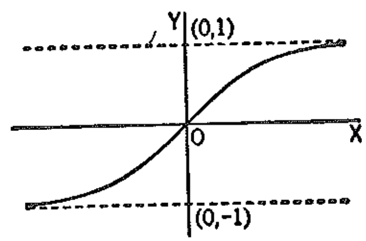
HYPERBOLIC SINE

$$y = \sinh x$$



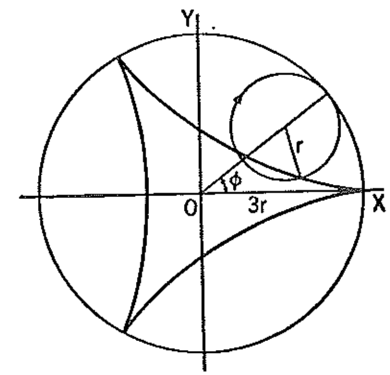
HYPERBOLIC COSINE

$$y = \cosh x$$



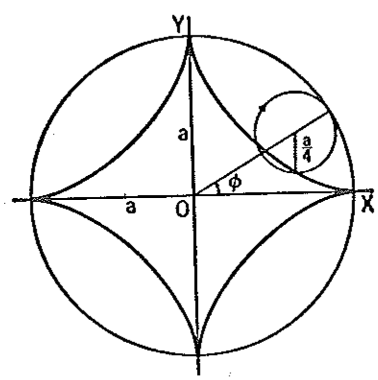
HYPERBOLIC TANGENT

$$y = \tanh x$$



HYPOCYCLOID OF THREE CUSPS

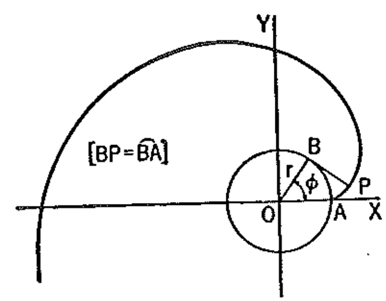
$$\begin{cases} x = 2r \cos \varphi + r \cos 2\varphi \\ y = 2r \sin \varphi - r \sin 2\varphi \end{cases}$$



HYPOCYCLOID OF FOUR CUSPS

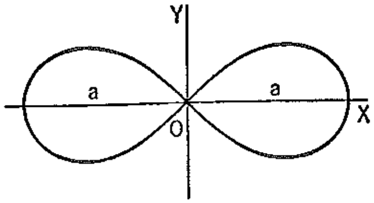
$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$x = a \cos^3 \varphi, \quad y = a \sin^3 \varphi$$



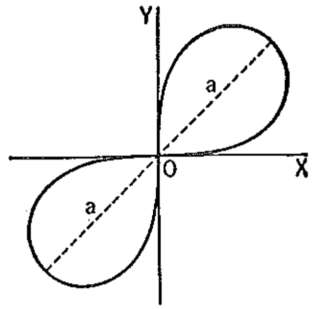
INVOLUTE OF CIRCLE

$$\begin{cases} x = r \cos \varphi + r\varphi \sin \varphi \\ y = r \sin \varphi - r\varphi \cos \varphi \end{cases}$$



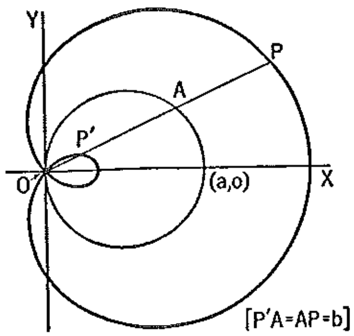
LEMNISCATE OF BERNOULLI

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$
$$\rho^2 = a^2 \cos 2\theta$$



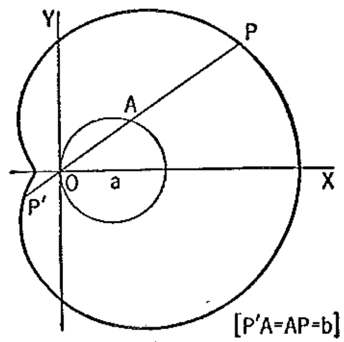
LEMNISCATE, TWO-LEAVED ROSE

$$(x^2 + y^2)^2 = 2a^2xy$$
$$\rho^2 = a^2 \sin 2\theta$$



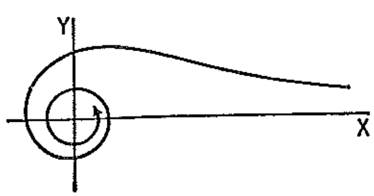
LIMAÇON OF PASCAL ($a > b$)

$$\rho = b + a \cos \theta$$



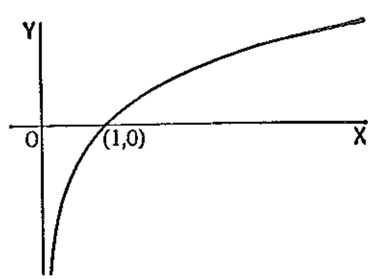
LIMAÇON OF PASCAL ($a < b$)

$$\rho = b + a \cos \theta$$



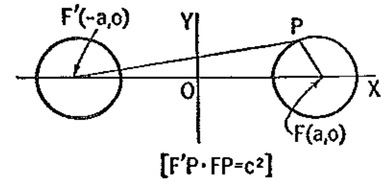
LITUUS

$$\rho^2\theta = a^2$$



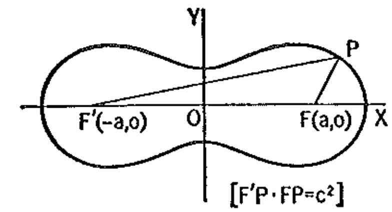
LOGARITHMIC CURVE

$$y = \log_e x$$



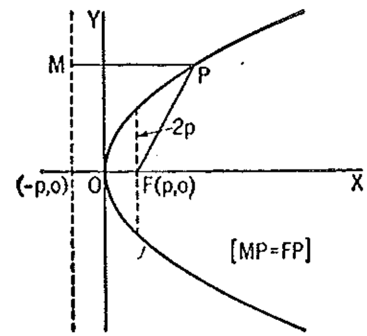
OVALS OF CASSINI ($a > c$)

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = c^4$$



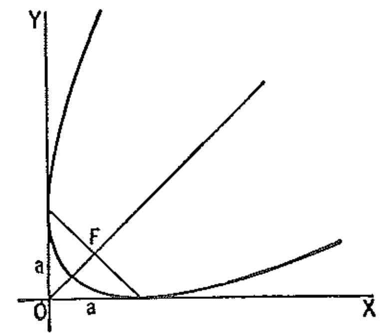
OVAL OF CASSINI ($a < c$)

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = c^4$$



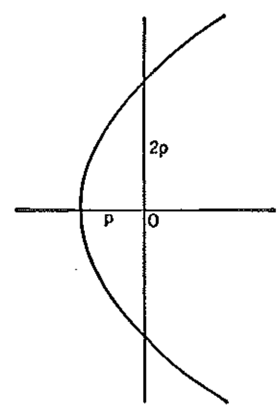
PARABOLA

$$y^2 = 4px$$



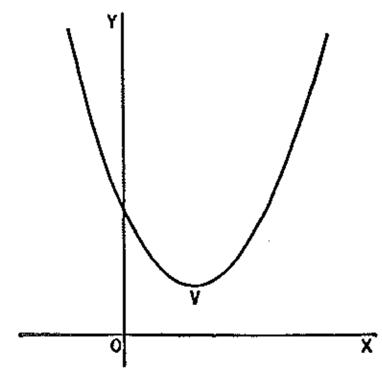
PARABOLA

$$\pm x^{1/2} \pm y^{1/2} = a^{1/2}$$



PARABOLA

$$\rho = \frac{2p}{1 - \cos \theta}$$

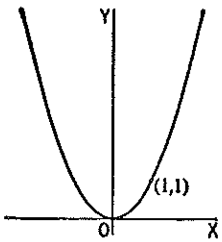


PARABOLA

$$y = ax^2 + bx + c, \quad a > o$$

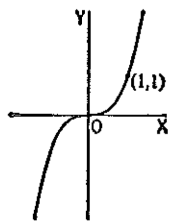
$$\left[\text{Abscissa of vertex} = -\frac{b}{2a}\right]$$

POWER FUNCTIONS



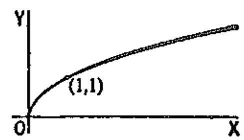
$y = x^2$

(Parabola)



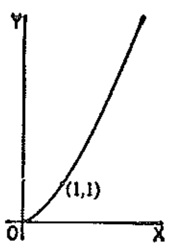
$y = x^3$

(Cubical Parabola)

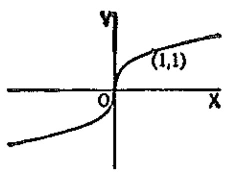


$y = x^{1/2}$

(Semiparabola)

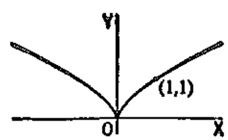


$y = x^{3/2}$

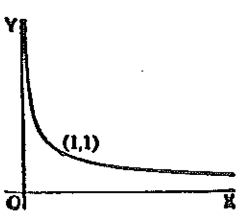


$y = x^{1/3}$

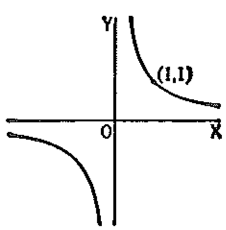
(Cubical Parabola)



$y = x^{2/3}$

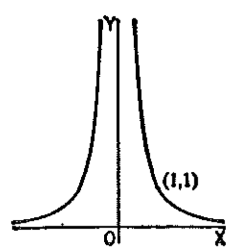


$y = x^{-1/2}$

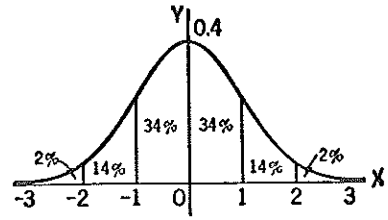


$y = x^{-1}$

(Equilateral Hyperbola)

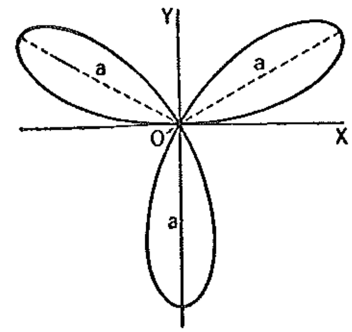


$y = x^{-2}$



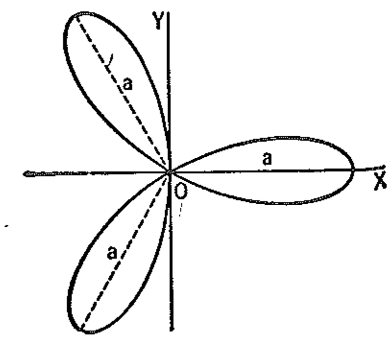
PROBABILITY CURVE

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



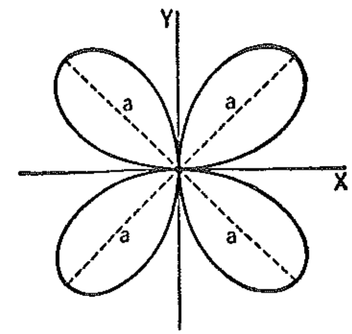
ROSE, THREE-LEAVED

$\rho = a \sin 3\theta$



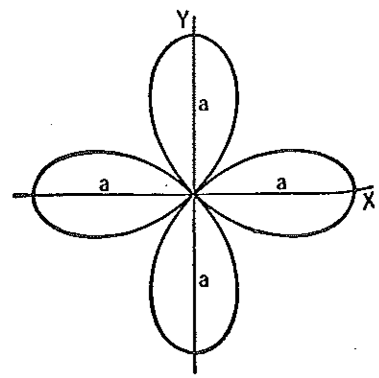
ROSE, THREE-LEAVED

$\rho = a \cos 3\theta$



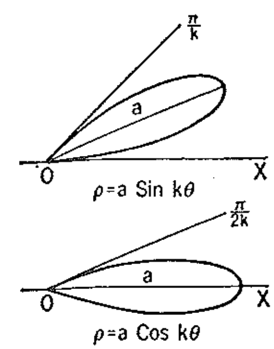
ROSE, FOUR-LEAVED

$\rho = a \sin 2\theta$



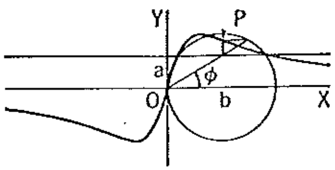
ROSE, FOUR-LEAVED

$\rho = a \cos 2\theta$



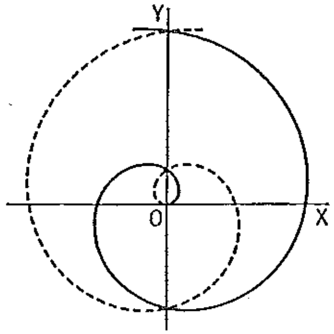
ROSE, n-LEAVED

If k is even, $n = 2k$.
If k is odd, $n = k$.
(The diagram shows the position of the first leaf.)



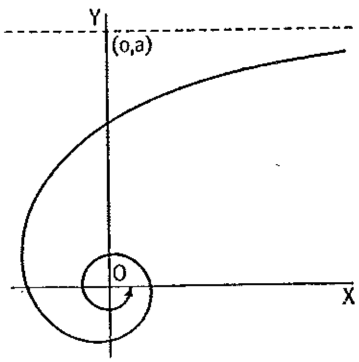
SERPENTINE

$$(a^2 + x^2)y = abx$$
$$x = a \operatorname{ctn} \varphi,\ y = b \sin \varphi \cos \varphi$$



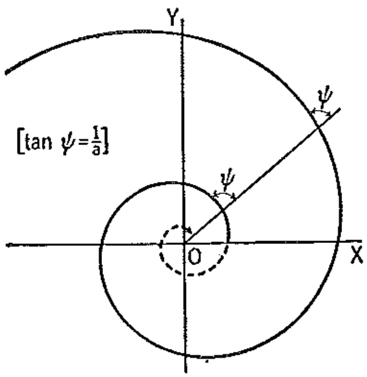
SPIRAL OF ARCHIMEDES

$$\rho = a\theta$$



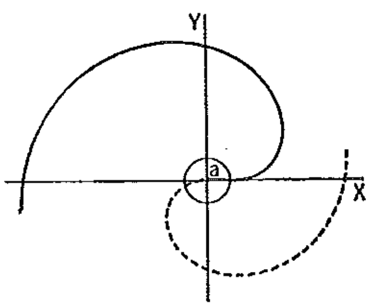
SPIRAL, HYPERBOLIC

$$\rho\theta = a$$



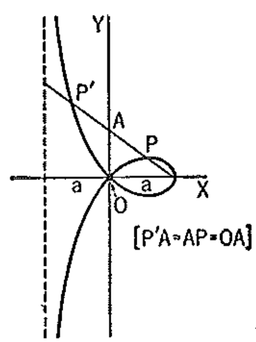
SPIRAL, LOGARITHMIC

$$\rho = e^{a\theta}, \quad \text{or} \quad \log \rho = a\theta$$



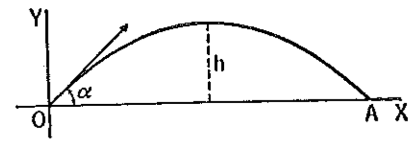
SPIRAL, PARABOLIC

$$(\rho - a)^2 = 4ac\theta$$



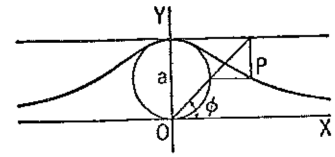
STROPHOID

$$y^2 = x^2 \frac{a - x}{a + x}$$
$$\rho = a \cos 2\theta \sec \theta$$



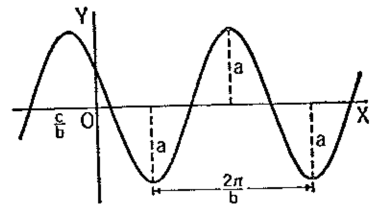
TRAJECTORY

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$$
$$x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$



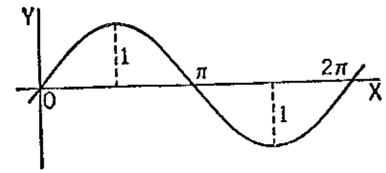
WITCH OF AGNESI

$$y = \frac{a^3}{(x^2 + a^2)}$$
$$x = a \operatorname{ctn} \varphi, \quad y = a \sin^2 \varphi$$



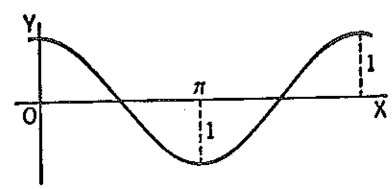
SINUSOID

$$y = a \sin (bx + c)$$



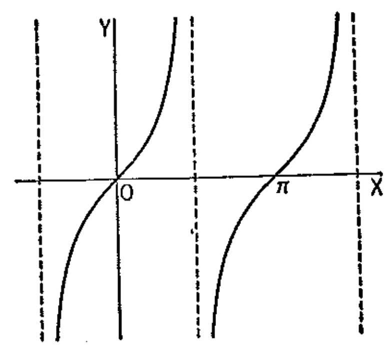
SINE CURVE

$$y = \sin x$$



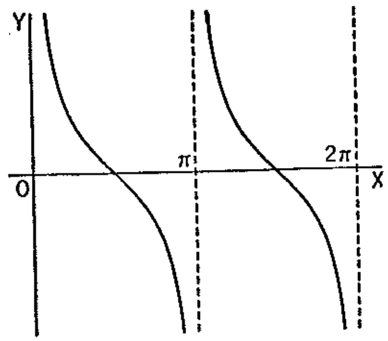
COSINE CURVE

$$y = \cos x$$



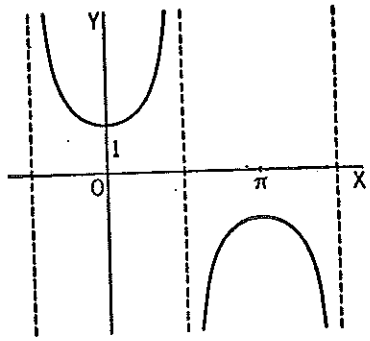
TANGENT CURVE

$$y = \tan x$$



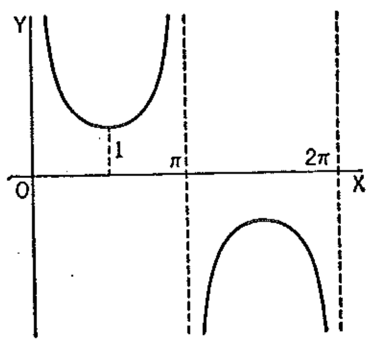
COTANGENT CURVE

$y = \operatorname{ctn} x$



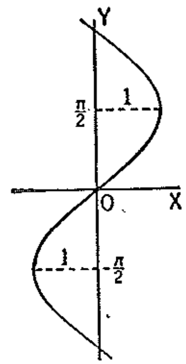
SECANT CURVE

$y = \sec x$



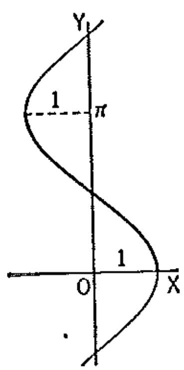
COSECANT CURVE

$y = \csc x$



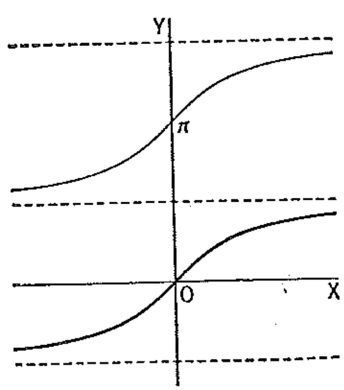
INVERSE SINE CURVE

$y = \arcsin x$



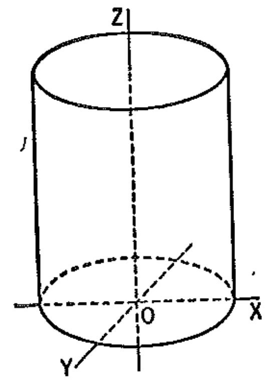
INVERSE COSINE CURVE

$y = \arccos x$



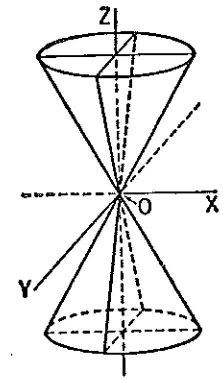
INVERSE TANGENT CURVE

$y = \arctan x$



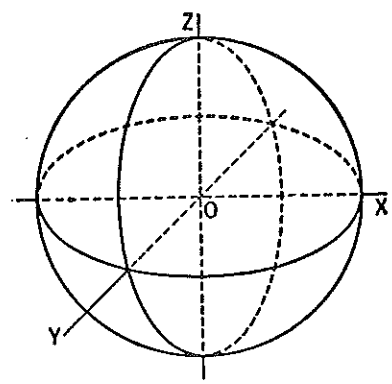
ELLIPTIC CYLINDER

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



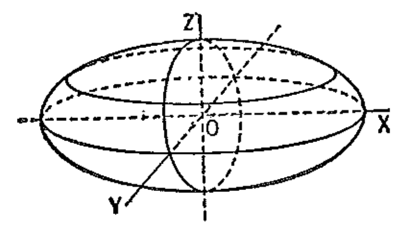
ELLIPTIC CONE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



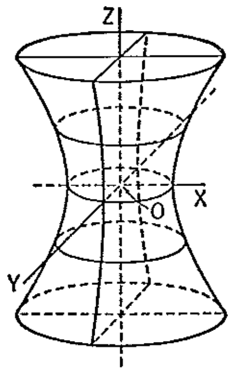
SPHERE

$$x^2 + y^2 + z^2 = a^2$$



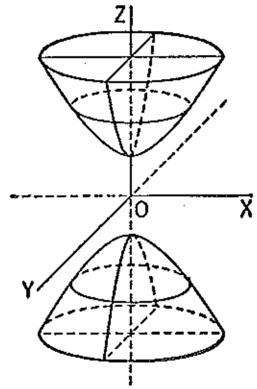
ELLIPSOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



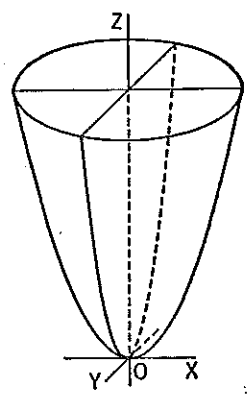
HYPERBOLOID OF ONE SHEET

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



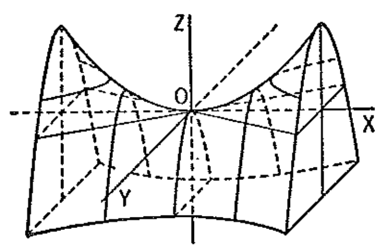
HYPERBOLOID OF TWO SHEETS

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



ELLIPTIC PARABOLOID

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}=cz$$



HYPERBOLIC PARABOLOID

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=cz$$

12 Derivatives

In these formulas u, v, w represent functions of x, and a, c, n represent constants. All arguments of the trigonometric functions are in radians.

1. $\frac{d}{dx}(c)=0$

2. $\frac{d}{dx}(x)=1$

3. $\frac{d}{dx}(cu)=c\frac{du}{dx}$

4. $\frac{d}{dx}(u+v-w)=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}$

5. $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

6. $\frac{d}{dx}(uvw)=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}$

7. $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

8. $\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$

9. $\frac{d}{dx}(\sqrt{u})=\frac{1}{2\sqrt{u}}\frac{du}{dx}$

10. $\frac{d}{dx}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{du}{dx}$

11. $\frac{d}{dx}f(u)=\frac{d}{du}f(u)\frac{du}{dx}$

12. $\frac{du}{dx}=\frac{1}{\frac{dx}{du}}$

13. $\frac{d}{dx}(\log_a u) = (\log_a e)\frac{1}{u}\frac{du}{dx}$

14. $\frac{d}{dx}(\log_e u) = \frac{1}{u}\frac{du}{dx}$

15. $\frac{d}{dx}(a^u) = (\log_e a)\, a^u \frac{du}{dx}$

16. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

17. $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + (\log_e u)\, u^v \frac{dv}{dx}$

18. $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

19. $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

20. $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$

21. $\frac{d}{dx}(\operatorname{ctn} u) = -\csc^2 u \frac{du}{dx}$

22. $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$

23. $\frac{d}{dx}(\csc u) = -\csc u \operatorname{ctn} u \frac{du}{dx}$

24. $\frac{d}{dx}(\operatorname{vers} u) = \sin u \frac{du}{dx}$

25. $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \left(-\frac{\pi}{2} \leqq \arcsin u \leqq \frac{\pi}{2}\right)$

26. $\frac{d}{dx}(\arccos u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (0 \leqq \arccos u \leqq \pi)$

27. $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx} \qquad \left(-\frac{\pi}{2} < \arctan u < \frac{\pi}{2}\right)$

28. $\frac{d}{dx}(\operatorname{arcctn} u) = -\frac{1}{1+u^2}\frac{du}{dx} \qquad (0 < \operatorname{arcctn} u < \pi)$

29. $\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

$\left(0 \leqq \operatorname{arcsec} u < \frac{\pi}{2}, \quad -\pi \leqq \operatorname{arcsec} u < -\frac{\pi}{2}\right)$

30. $\frac{d}{dx}(\operatorname{arccsc} u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

$\left(0 < \operatorname{arccsc} u \leqq \frac{\pi}{2}, \quad -\pi < \operatorname{arccsc} u \leqq -\frac{\pi}{2}\right)$

31. $\frac{d}{dx}(\operatorname{arcvers} u) = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx} \qquad (0 \leqq \operatorname{arcvers} u \leqq \pi)$

32. $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

33. $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$

34. $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$

35. $\frac{d}{dx}(\operatorname{ctnh} u) = -\operatorname{csch}^2 u \frac{du}{dx}$

36. $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$

37. $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \operatorname{ctnh} u \frac{du}{dx}$

38. $\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}[\log_e(u + \sqrt{u^2+1})] = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

39. $\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}[\log_e(u + \sqrt{u^2-1})] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \qquad (u > 1)$

40. $\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log_e \frac{1+u}{1-u}\right] = \frac{1}{1-u^2}\frac{du}{dx} \qquad (u^2 < 1)$

41. $\frac{d}{dx}(\operatorname{ctnh}^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log_e \frac{u+1}{u-1}\right] = \frac{1}{1-u^2}\frac{du}{dx} \qquad (u^2 > 1)$

42. $\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{d}{dx}\left[\log_e \frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$

$(0 < u < 1)$

43. $\frac{d}{dx}(\operatorname{csch}^{-1} u) = \frac{d}{dx}\left[\log_e \frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{u\sqrt{1+u^2}}\frac{du}{dx}$

$(u > 0)$

13 Indefinite Integrals

A constant of integration should be added to each of the following formulas. All arguments of the trigonometric functions are in radians and all logarithms are in the natural system.

FUNDAMENTAL FORMS

1. $\int df(x) = f(x)$

2. $\int dx = x$

3. $\int af(x)\,dx = a\int f(x)\,dx$

4. $\int [u(x) \pm v(x)]\,dx = \int u(x)\,dx \pm \int v(x)\,dx$

5. $\int x^m\,dx = \dfrac{x^{m+1}}{m+1}$ $\qquad (m \neq -1)$

6. $\int \dfrac{dx}{x} = \log|x|$

7. $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$

8. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a}$

9. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right|$

10. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{a-x}{a+x}\right|$

11. $\int u\,dv = uv - \int v\,du$

12. $\int u\dfrac{dv}{dx}\,dx = uv - \int v\dfrac{du}{dx}\,dx$

13. $\int f(y)\,dx = \int \dfrac{f(y)\,dy}{\dfrac{dy}{dx}}$

FORMS INVOLVING $ax + b$

14. $\int (ax+b)^m\,dx = \dfrac{(ax+b)^{m+1}}{a(m+1)}$ $\qquad (m \neq -1)$

15. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\log|ax+b|$

16. $\int \dfrac{dx}{(ax+b)^2} = -\dfrac{1}{a(ax+b)}$

17. $\int \dfrac{dx}{(ax+b)^3} = -\dfrac{1}{2a(ax+b)^2}$

18. $\int x(ax+b)^m\,dx = \dfrac{(ax+b)^{m+2}}{a^2(m+2)} - \dfrac{b(ax+b)^{m+1}}{a^2(m+1)}$, $\quad (m \neq -1, -2)$

19. $\int \dfrac{x\,dx}{ax+b} = \dfrac{x}{a} - \dfrac{b}{a^2}\log|ax+b|$

20. $\int \dfrac{x\,dx}{(ax+b)^2} = \dfrac{b}{a^2(ax+b)} + \dfrac{1}{a^2}\log|ax+b|$

21. $\int \dfrac{x\,dx}{(ax+b)^3} = \dfrac{b}{2a^2(ax+b)^2} - \dfrac{1}{a^2(ax+b)}$

22. $\int x^2(ax+b)^m\,dx$

$$= \frac{1}{a^3}\left[\frac{(ax+b)^{m+3}}{m+3} - \frac{2b(ax+b)^{m+2}}{m+2} + \frac{b^2(ax+b)^{m+1}}{m+1}\right]$$

$(m \neq -1, -2, -3)$

23. $\int \dfrac{x^2\,dx}{ax+b} = \dfrac{1}{a^3}\left[\tfrac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\log|ax+b|\right]$

24. $\int \dfrac{x^2\,dx}{(ax+b)^2} = \dfrac{1}{a^3}\left[(ax+b) - \dfrac{b^2}{ax+b} - 2b\log|ax+b|\right]$

25. $\int \dfrac{x^2\,dx}{(ax+b)^3} = \dfrac{1}{a^3}\left[\log|ax+b| + \dfrac{2b}{ax+b} - \dfrac{b^2}{2(ax+b)^2}\right]$

26. $$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\log\left|\frac{x}{ax+b}\right|$$

27. $$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\log\left|\frac{ax+b}{x}\right|$$

28. $$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2}\log\left|\frac{ax+b}{x}\right|$$

29. $$\int \frac{dx}{x^2(ax+b)^2} = -\frac{2ax+b}{b^2x(ax+b)} + \frac{2a}{b^3}\log\left|\frac{ax+b}{x}\right|$$

30. $$\int x^m(ax+b)^n\,dx$$
$$= \frac{1}{a(m+n+1)}\left[x^m(ax+b)^{n+1} - mb\int x^{m-1}(ax+b)^n\,dx\right]$$
$$= \frac{1}{m+n+1}\left[x^{m+1}(ax+b)^n + nb\int x^m(ax+b)^{n-1}\,dx\right]$$
$$(m > 0, \quad m+n+1 \neq 0)$$

If n is a positive integer, this form may be integrated term by term after expanding $(ax+b)^n$ by the binomial theorem.

31. $$\int x^m(ax+b)^n\,dx = \frac{1}{a^{m+1}}\int u^n(u-b)^m\,du \qquad (u = ax+b)$$

See the note after Formula 30.

32. $$\int \frac{x^m\,dx}{(ax+b)^n} = \frac{1}{a^{m+1}}\int \frac{(u-b)^m\,du}{u^n} \qquad (u = ax+b)$$

See the note after Formula 30.

33. $$\int \frac{dx}{x^m(ax+b)^n} = -\frac{1}{b^{m+n-1}}\int \frac{(v-a)^{m+n-2}\,dv}{v^n} \qquad \left(v = \frac{ax+b}{x}\right)$$

See the note after Formula 30.

FORMS INVOLVING $ax + b$ AND $cx + d$ $(bc \neq ad)$

34. $$\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\log\left|\frac{cx+d}{ax+b}\right|$$

35. $$\int \frac{x\,dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\left[\frac{b}{a}\log|ax+b| - \frac{d}{c}\log|cx+d|\right]$$

36. $$\int \frac{dx}{(ax+b)^2(cx+d)} = \frac{1}{bc-ad}\left[\frac{1}{ax+b} + \frac{c}{bc-ad}\log\left|\frac{cx+d}{ax+b}\right|\right]$$

37. $$\int \frac{x\,dx}{(ax+b)^2(cx+d)}$$
$$= \frac{1}{bc-ad}\left[-\frac{b}{a(ax+b)} - \frac{d}{bc-ad}\log\left|\frac{cx+d}{ax+b}\right|\right]$$

38. $$\int \frac{x^2\,dx}{(ax+b)^2(cx+d)} = \frac{b^2}{a^2(bc-ad)(ax+b)}$$
$$+ \frac{1}{(bc-ad)^2}\left[\frac{d^2}{c}\log|cx+d| + \frac{b(bc-2ad)}{a^2}\log|ax+b|\right]$$

39. $$\int \frac{ax+b}{cx+d}\,dx = \frac{ax}{c} + \frac{bc-ad}{c^2}\log|cx+d|$$

40. $$\int (ax+b)^m(cx+d)^n\,dx = \frac{1}{a(m+n+1)}\left[(ax+b)^{m+1}(cx+d)^n\right.$$
$$\left. - n(bc-ad)\int (ax+b)^m(cx+d)^{n-1}\,dx\right]$$

41. $$\int \frac{(ax+b)^m\,dx}{(cx+d)^n}$$
$$= -\frac{1}{(n-1)(bc-ad)}\left[\frac{(ax+b)^{m+1}}{(cx+d)^{n-1}} + a(n-m-2)\int \frac{(ax+b)^m\,dx}{(cx+d)^{n-1}}\right]$$
$$= -\frac{1}{c(n-m-1)}\left[\frac{(ax+b)^m}{(cx+d)^{n-1}} + m(bc-ad)\int \frac{(ax+b)^{m-1}\,dx}{(cx+d)^n}\right]$$
$$= -\frac{1}{(n-1)c}\left[\frac{(ax+b)^m}{(cx+d)^{n-1}} - ma\int \frac{(ax+b)^{m-1}\,dx}{(cx+d)^{n-1}}\right]$$

42. $$\int \frac{dx}{(ax+b)^m(cx+d)^n} = \frac{-1}{(n-1)(bc-ad)}\left[\frac{1}{(ax+b)^{m-1}(cx+d)^{n-1}}\right.$$
$$\left. + a(m+n-2)\int \frac{dx}{(ax+b)^m(cx+d)^{n-1}}\right]$$

FORMS INVOLVING $ax^2 + c$

43. $$\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}}\arctan\left(x\sqrt{\frac{a}{c}}\right) \qquad (a > 0, \quad c > 0)$$

44. $$\int \frac{dx}{ax^2+c} = \frac{1}{2\sqrt{-ac}}\log\left|\frac{x\sqrt{a} - \sqrt{-c}}{x\sqrt{a} + \sqrt{-c}}\right| \qquad (a > 0, \quad c < 0)$$

45. $$\int \frac{dx}{ax^2+c} = \frac{1}{2\sqrt{-ac}}\log\left|\frac{\sqrt{c} + x\sqrt{-a}}{\sqrt{c} - x\sqrt{-a}}\right| \qquad (a < 0, \quad c > 0)$$

46. $$\int \frac{x\,dx}{ax^2+c} = \frac{1}{2a}\log|ax^2+c|$$

47. $$\int \frac{x^2\,dx}{ax^2+c} = \frac{x}{a} - \frac{c}{a}\int \frac{dx}{ax^2+c}$$

48. $$\int \frac{x^m\,dx}{ax^2+c} = \frac{x^{m-1}}{a(m-1)} - \frac{c}{a}\int \frac{x^{m-2}\,dx}{ax^2+c} \qquad (m \neq 1)$$

49. $$\int \frac{dx}{x(ax^2+c)} = \frac{1}{2c}\log\left|\frac{ax^2}{ax^2+c}\right|$$

50. $$\int \frac{dx}{x^2(ax^2+c)} = -\frac{1}{cx} - \frac{a}{c}\int \frac{dx}{ax^2+c}$$

51. $$\int \frac{dx}{x^m(ax^2+c)} = -\frac{1}{c(m-1)x^{m-1}} - \frac{a}{c}\int \frac{dx}{x^{m-2}(ax^2+c)} \qquad (m \neq 1)$$

52. $$\int \frac{dx}{(ax^2+c)^m} = \frac{1}{2(m-1)c}\cdot\frac{x}{(ax^2+c)^{m-1}} + \frac{2m-3}{2(m-1)c}\int \frac{dx}{(ax^2+c)^{m-1}} \qquad (m \neq 1)$$

53. $$\int \frac{x\,dx}{(ax^2+c)^m} = -\frac{1}{2a(m-1)(ax^2+c)^{m-1}} \qquad (m \neq 1)$$

54. $$\int \frac{x^2\,dx}{(ax^2+c)^m} = -\frac{x}{2a(m-1)(ax^2+c)^{m-1}} + \frac{1}{2a(m-1)}\int \frac{dx}{(ax^2+c)^{m-1}} \qquad (m \neq 1)$$

55. $$\int \frac{dx}{x(ax^2+c)^m} = \frac{1}{2c(m-1)(ax^2+c)^{m-1}} + \frac{1}{c}\int \frac{dx}{x(ax^2+c)^{m-1}} \qquad (m \neq 1)$$

56. $$\int \frac{dx}{x^2(ax^2+c)^m} = \frac{1}{c}\int \frac{dx}{x^2(ax^2+c)^{m-1}} - \frac{a}{c}\int \frac{dx}{(ax^2+c)^m}$$

FORMS INVOLVING $ax^2 + bx + c$

Let $X = ax^2 + bx + c$, $D = b^2 - 4ac$.

57. $$\int X^m\,dx = \frac{1}{2a(2m+1)}\left[(2ax+b)X^m - Dm\int X^{m-1}\,dx\right]$$

58. $$\int \frac{dx}{X} = \frac{1}{\sqrt{D}}\log\left|\frac{2ax+b-\sqrt{D}}{2ax+b+\sqrt{D}}\right| \qquad (D > 0)$$

59. $$\int \frac{dx}{X} = \frac{2}{\sqrt{-D}}\arctan\frac{2ax+b}{\sqrt{-D}} \qquad (D < 0)$$

60. $$\int \frac{dx}{X} = -\frac{2}{2ax+b} \qquad (D = 0)$$

61. $$\int \frac{x\,dx}{X} = \frac{1}{2a}\log|X| - \frac{b}{2a}\int \frac{dx}{X}$$

62. $$\int \frac{dx}{xX} = \frac{1}{2c}\log\frac{x^2}{|X|} - \frac{b}{2c}\int \frac{dx}{X}$$

63. $$\int \frac{x^m\,dx}{X} = \frac{x^{m-1}}{a(m-1)} - \frac{c}{a}\int \frac{x^{m-2}\,dx}{X} - \frac{b}{a}\int \frac{x^{m-1}\,dx}{X} \qquad (m \neq 1)$$

64. $$\int \frac{dx}{x^m X} = -\frac{1}{(m-1)cx^{m-1}} - \frac{b}{c}\int \frac{dx}{x^{m-1}X} - \frac{a}{c}\int \frac{dx}{x^{m-2}X} \qquad (m \neq 1)$$

65. $$\int \frac{dx}{X^m} = -\frac{2ax+b}{(m-1)DX^{m-1}} - \frac{2(2m-3)a}{(m-1)D}\int \frac{dx}{X^{m-1}} \qquad (m \neq 1,\quad D \neq 0)$$

66. $$\int \frac{x\,dx}{X^m} = \frac{2c+bx}{(m-1)DX^{m-1}} + \frac{b(2m-3)}{(m-1)D}\int \frac{dx}{X^{m-1}} \qquad (m \neq 1,\quad D \neq 0)$$

67. $$\int \frac{dx}{xX^m} = \frac{1}{2c(m-1)X^{m-1}} - \frac{b}{2c}\int \frac{dx}{X^m} + \frac{1}{c}\int \frac{dx}{xX^{m-1}} \qquad (m \neq 1)$$

68. $$\int \frac{x^m\,dx}{X^n} = -\frac{x^{m-1}}{a(2n-m-1)X^{n-1}} - \frac{n-m}{2n-m-1}\cdot\frac{b}{a}\int \frac{x^{m-1}\,dx}{X^n} + \frac{m-1}{2n-m-1}\cdot\frac{c}{a}\int \frac{x^{m-2}\,dx}{X^n}$$

69. $$\int \frac{dx}{x^m X^n} = -\frac{1}{(m-1)cx^{m-1}X^{n-1}} - \frac{n+m-2}{m-1}\cdot\frac{b}{c}\int \frac{dx}{x^{m-1}X^n} - \frac{2n+m-3}{m-1}\cdot\frac{a}{c}\int \frac{dx}{x^{m-2}X^n} \qquad (m \neq 1)$$

FORMS INVOLVING $\sqrt{ax+b}$, WHERE $ax + b > 0$

70. $$\int \sqrt{ax+b}\,dx = \frac{2}{3a}\sqrt{(ax+b)^3}$$

71. $$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

72. $$\int x^2\sqrt{ax+b}\,dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$$

73. $$\int x^m\sqrt{ax+b}\,dx = \frac{2}{a(2m+3)}\left[x^m\sqrt{(ax+b)^3} - mb\int x^{m-1}\sqrt{ax+b}\,dx\right]$$

74. $$\int \frac{\sqrt{ax+b}\,dx}{x} = 2\sqrt{ax+b} + \sqrt{b}\log\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| \qquad (b > 0)$$

75. $$\int \frac{\sqrt{ax+b}\,dx}{x} = 2\sqrt{ax+b} - 2\sqrt{-b}\arctan\sqrt{\frac{ax+b}{-b}} \qquad (b < 0)$$

76. $$\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int\frac{dx}{x\sqrt{ax+b}}$$

77. $$\int \frac{\sqrt{ax+b}\,dx}{x^m} = -\frac{1}{(m-1)b}\left[\frac{\sqrt{(ax+b)^3}}{x^{m-1}} + \frac{(2m-5)a}{2}\int\frac{\sqrt{ax+b}\,dx}{x^{m-1}}\right] \qquad (m \neq 1)$$

78. $$\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$$

79. $$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)\sqrt{ax+b}}{3a^2}$$

80. $$\int \frac{x^2\,dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)\sqrt{ax+b}}{15a^3}$$

81. $$\int \frac{x^m\,dx}{\sqrt{ax+b}} = \frac{2}{a(2m+1)}\left[x^m\sqrt{ax+b} - mb\int\frac{x^{m-1}\,dx}{\sqrt{ax+b}}\right] \qquad (m \neq -\tfrac{1}{2})$$

82. $$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\log\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| \qquad (b > 0)$$

83. $$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}} \qquad (b < 0)$$

84. $$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int\frac{dx}{x\sqrt{ax+b}}$$

85. $$\int \frac{dx}{x^m\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{(2m-2)b}\int\frac{dx}{x^{m-1}\sqrt{ax+b}} \qquad (m \neq 1)$$

86. $$\int (ax+b)^{\pm\frac{m}{2}}\,dx = \frac{2(ax+b)^{\frac{2\pm m}{2}}}{a(2\pm m)}$$

87. $$\int x(ax+b)^{\pm\frac{m}{2}}\,dx = \frac{2}{a^2}\left[\frac{(ax+b)^{\frac{4\pm m}{2}}}{4\pm m} - \frac{b(ax+b)^{\frac{2\pm m}{2}}}{2\pm m}\right]$$

88. $$\int \frac{(ax+b)^{\frac{m}{2}}\,dx}{x} = a\int (ax+b)^{\frac{m-2}{2}}\,dx + b\int\frac{(ax+b)^{\frac{m-2}{2}}\,dx}{x}$$

89. $$\int \frac{dx}{x(ax+b)^{\frac{m}{2}}} = \frac{1}{b}\int\frac{dx}{x(ax+b)^{\frac{m-2}{2}}} - \frac{a}{b}\int\frac{dx}{(ax+b)^{\frac{m}{2}}}$$

90. $$\int F(x, \sqrt{ax+b})\,dx = \frac{2}{a}\int F\left(\frac{u^2-b}{a}, u\right)u\,du \qquad (ax+b = u^2)$$

91. $$\int \frac{\sqrt{ax+b}\,dx}{cx+d} = \frac{2\sqrt{ax+b}}{c} + \frac{1}{c}\sqrt{\frac{bc-ad}{c}}\log\left|\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}\right| \qquad (c > 0, \quad bc > ad)$$

92. $$\int \frac{\sqrt{ax+b}\,dx}{cx+d} = \frac{2\sqrt{ax+b}}{c} - \frac{2}{c}\sqrt{\frac{ad-bc}{c}}\arctan\sqrt{\frac{c(ax+b)}{ad-bc}} \qquad (c > 0, \quad bc < ad)$$

93. $$\int \frac{(cx+d)\,dx}{\sqrt{ax+b}} = \frac{2}{3a^2}(3ad-2bc+acx)\sqrt{ax+b}$$

94. $$\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{2}{\sqrt{c}\sqrt{ad-bc}}\arctan\sqrt{\frac{c(ax+b)}{ad-bc}} \qquad (c > 0, \quad bc < ad)$$

95. $$\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{1}{\sqrt{c}\sqrt{bc-ad}}\log\left|\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}\right| \qquad (c > 0, \quad bc > ad)$$

96. $$\int \sqrt{ax+b}\sqrt{cx+d}\,dx = \int\sqrt{acx^2+(ad+bc)x+bd}\,dx$$

(See Formula 186)

97. $$\int \frac{dx}{\sqrt{ax+b}\sqrt{cx+d}} = \int\frac{dx}{\sqrt{acx^2+(ad+bc)x+bd}}$$

(See Formulas 191, 192)

98. $$\int \frac{\sqrt{ax+b}\,dx}{\sqrt{cx+d}} = \int\frac{(ax+b)\,dx}{\sqrt{acx^2+(ad+bc)x+bd}}$$

(See Formula 201)

FORMS INVOLVING $\sqrt{a^2+x^2}$ OR $\sqrt{(a^2+x^2)^3}$

99. $$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x+\sqrt{a^2+x^2})$$

100. $$\int x\sqrt{a^2+x^2}\,dx = \tfrac{1}{3}\sqrt{(a^2+x^2)^3}$$

101. $$\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{4}\sqrt{(a^2+x^2)^3} - \frac{a^2x}{8}\sqrt{a^2+x^2} - \frac{a^4}{8}\log\left(x+\sqrt{a^2+x^2}\right)$$

102. $$\int x^3\sqrt{a^2+x^2}\,dx = \left(\tfrac{1}{5}x^2 - \tfrac{2}{15}a^2\right)\sqrt{(a^2+x^2)^3}$$

103. $$\int \frac{\sqrt{a^2+x^2}\,dx}{x} = \sqrt{a^2+x^2} - a\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

104. $$\int \frac{\sqrt{a^2+x^2}\,dx}{x^2} = -\frac{\sqrt{a^2+x^2}}{x} + \log\left(x+\sqrt{a^2+x^2}\right)$$

105. $$\int \frac{\sqrt{a^2+x^2}\,dx}{x^3} = -\frac{\sqrt{a^2+x^2}}{2x^2} - \frac{1}{2a}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

106. $$\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left(x+\sqrt{a^2+x^2}\right)$$

107. $$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}$$

108. $$\int \frac{x^2\,dx}{\sqrt{a^2+x^2}} = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$$

109. $$\int \frac{x^3\,dx}{\sqrt{a^2+x^2}} = \tfrac{1}{3}\sqrt{(a^2+x^2)^3} - a^2\sqrt{a^2+x^2}$$

110. $$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

111. $$\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x}$$

112. $$\int \frac{dx}{x^3\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

113. $$\int \sqrt{(a^2+x^2)^3}\,dx$$
$$= \frac{x}{4}\sqrt{(a^2+x^2)^3} + \frac{3a^2x}{8}\sqrt{a^2+x^2} + \frac{3a^4}{8}\log\left(x+\sqrt{a^2+x^2}\right)$$

114. $$\int x\sqrt{(a^2+x^2)^3}\,dx = \tfrac{1}{5}\sqrt{(a^2+x^2)^5}$$

115. $$\int x^2\sqrt{(a^2+x^2)^3}\,dx = \frac{x}{6}\sqrt{(a^2+x^2)^5} - \frac{a^2x}{24}\sqrt{(a^2+x^2)^3} - \frac{a^4x}{16}\sqrt{a^2+x^2}$$
$$- \frac{a^6}{16}\log\left(x+\sqrt{a^2+x^2}\right)$$

116. $$\int x^3\sqrt{(a^2+x^2)^3}\,dx = \tfrac{1}{7}\sqrt{(a^2+x^2)^7} - \frac{a^2}{5}\sqrt{(a^2+x^2)^5}$$

117. $$\int \frac{\sqrt{(a^2+x^2)^3}\,dx}{x} = \tfrac{1}{3}\sqrt{(a^2+x^2)^3} + a^2\sqrt{a^2+x^2} - a^3\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

118. $$\int \frac{\sqrt{(a^2+x^2)^3}\,dx}{x^2}$$
$$= -\frac{1}{x}\sqrt{(a^2+x^2)^3} + \frac{3x}{2}\sqrt{a^2+x^2} + \frac{3a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$$

119. $$\int \frac{\sqrt{(a^2+x^2)^3}\,dx}{x^3} = -\frac{\sqrt{(a^2+x^2)^3}}{2x^2} + \tfrac{3}{2}\sqrt{a^2+x^2} - \frac{3a}{2}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

120. $$\int \frac{dx}{\sqrt{(a^2+x^2)^3}} = \frac{x}{a^2\sqrt{a^2+x^2}}$$

121. $$\int \frac{x\,dx}{\sqrt{(a^2+x^2)^3}} = \frac{-1}{\sqrt{a^2+x^2}}$$

122. $$\int \frac{x^2\,dx}{\sqrt{(a^2+x^2)^3}} = -\frac{x}{\sqrt{a^2+x^2}} + \log\left(x+\sqrt{x^2+a^2}\right)$$

123. $$\int \frac{x^3\,dx}{\sqrt{(a^2+x^2)^3}} = \sqrt{a^2+x^2} + \frac{a^2}{\sqrt{a^2+x^2}}$$

124. $$\int \frac{dx}{x\sqrt{(a^2+x^2)^3}} = \frac{1}{a^2\sqrt{a^2+x^2}} - \frac{1}{a^3}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

125. $$\int \frac{dx}{x^2\sqrt{(a^2+x^2)^3}} = -\frac{1}{a^4}\left[\frac{\sqrt{a^2+x^2}}{x} + \frac{x}{\sqrt{a^2+x^2}}\right]$$

126. $$\int \frac{dx}{x^3\sqrt{(a^2+x^2)^3}}$$
$$= -\frac{1}{2a^2x^2\sqrt{a^2+x^2}} - \frac{3}{2a^4\sqrt{a^2+x^2}} + \frac{3}{2a^5}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

127. $$\int F\left(x, \sqrt{a^2+x^2}\right)dx = a\int F(a\tan z, a\sec z)\sec^2 z\,dz$$
$$(x = a\tan z)$$

FORMS INVOLVING $\sqrt{a^2-x^2}$ OR $\sqrt{(a^2-x^2)^3}$, WHERE $a^2 > x^2$, $a > 0$

128. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a}\right)$

129. $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3}$

130. $\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right)$

131. $\int x^3\sqrt{a^2-x^2}\,dx = (-\frac{1}{5}x^2 - \frac{2}{15}a^2)\sqrt{(a^2-x^2)^3}$

132. $\int \frac{\sqrt{a^2-x^2}\,dx}{x} = \sqrt{a^2-x^2} - a\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

133. $\int \frac{\sqrt{a^2-x^2}\,dx}{x^2} = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a}$

134. $\int \frac{\sqrt{a^2-x^2}\,dx}{x^3} = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

135. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$

136. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$

137. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$

138. $\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \frac{1}{3}\sqrt{(a^2-x^2)^3} - a^2\sqrt{a^2-x^2}$

139. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

140. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$

141. $\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

142. $\int \sqrt{(a^2-x^2)^3}\,dx = \frac{1}{4}x\sqrt{(a^2-x^2)^3} + \frac{3a^2x}{8}\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}$

143. $\int x\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2-x^2)^5}$

144. $\int x^2\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{6}x\sqrt{(a^2-x^2)^5} + \frac{a^2x}{24}\sqrt{(a^2-x^2)^3}$

$$+\frac{a^4x}{16}\sqrt{a^2-x^2} + \frac{a^6}{16}\arcsin\frac{x}{a}$$

145. $\int x^3\sqrt{(a^2-x^2)^3}\,dx = \frac{1}{7}\sqrt{(a^2-x^2)^7} - \frac{a^2}{5}\sqrt{(a^2-x^2)^5}$

146. $\int \frac{\sqrt{(a^2-x^2)^3}\,dx}{x} = \frac{1}{3}\sqrt{(a^2-x^2)^3} + a^2\sqrt{a^2-x^2} - a^3\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

147. $\int \frac{\sqrt{(a^2-x^2)^3}\,dx}{x^2} = -\frac{\sqrt{(a^2-x^2)^3}}{x} - \frac{3x}{2}\sqrt{a^2-x^2} - \frac{3a^2}{2}\arcsin\frac{x}{a}$

148. $\int \frac{\sqrt{(a^2-x^2)^3}\,dx}{x^3} = -\frac{\sqrt{(a^2-x^2)^3}}{2x^2} - \frac{3}{2}\sqrt{a^2-x^2} + \frac{3a}{2}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

149. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}$

150. $\int \frac{x\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{1}{\sqrt{a^2-x^2}}$

151. $\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a}$

152. $\int \frac{x^3\,dx}{\sqrt{(a^2-x^2)^3}} = \sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}$

153. $\int \frac{dx}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

154. $\int \frac{dx}{x^2\sqrt{(a^2-x^2)^3}} = \frac{1}{a^4}\left[-\frac{\sqrt{a^2-x^2}}{x} + \frac{x}{\sqrt{a^2-x^2}}\right]$

155. $\int \frac{dx}{x^3\sqrt{(a^2-x^2)^3}}$

$$= -\frac{1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

156. $\int F(x, \sqrt{a^2-x^2})\,dx = a\int F(a\sin z, a\cos z)\cos z\,dz$

$(x = a\sin z)$

FORMS INVOLVING $\sqrt{x^2-a^2}$ OR $\sqrt{(x^2-a^2)^3}$, WHERE $a^2 < x^2$, $a > 0$

157. $$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|$$

158. $$\int x\sqrt{x^2-a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2-a^2)^3}$$

159. $$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{4}\sqrt{(x^2-a^2)^3} + \frac{a^2x}{8}\sqrt{x^2-a^2} - \frac{a^4}{8}\log|x+\sqrt{x^2-a^2}|$$

160. $$\int x^3\sqrt{x^2-a^2}\,dx = \frac{1}{5}\sqrt{(x^2-a^2)^5} + \frac{a^2}{3}\sqrt{(x^2-a^2)^3}$$

161. $$\int \frac{\sqrt{x^2-a^2}\,dx}{x} = \sqrt{x^2-a^2} - a\,\mathrm{arccos}\,\frac{a}{x}$$

162. $$\int \frac{\sqrt{x^2-a^2}\,dx}{x^2} = -\frac{1}{x}\sqrt{x^2-a^2} + \log|x+\sqrt{x^2-a^2}|$$

163. $$\int \frac{\sqrt{x^2-a^2}\,dx}{x^3} = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\,\mathrm{arccos}\,\frac{a}{x}$$

164. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|$$

165. $$\int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$$

166. $$\int \frac{x^2\,dx}{\sqrt{x^2-a^2}} = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|$$

167. $$\int \frac{x^3\,dx}{\sqrt{x^2-a^2}} = \frac{1}{3}\sqrt{(x^2-a^2)^3} + a^2\sqrt{x^2-a^2}$$

168. $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\,\mathrm{arccos}\,\frac{a}{x}$$

169. $$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}$$

170. $$\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\,\mathrm{arccos}\,\frac{a}{x}$$

171. $$\int \sqrt{(x^2-a^2)^3}\,dx$$
$$= \frac{x}{4}\sqrt{(x^2-a^2)^3} - \frac{3a^2x}{8}\sqrt{x^2-a^2} + \frac{3a^4}{8}\log|x+\sqrt{x^2-a^2}|$$

172. $$\int x\sqrt{(x^2-a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2-a^2)^5}$$

173. $$\int x^2\sqrt{(x^2-a^2)^3}\,dx = \frac{x}{6}\sqrt{(x^2-a^2)^5} + \frac{a^2x}{24}\sqrt{(x^2-a^2)^3}$$
$$- \frac{a^4x}{16}\sqrt{x^2-a^2} + \frac{a^6}{16}\log|x+\sqrt{x^2-a^2}|$$

174. $$\int x^3\sqrt{(x^2-a^2)^3}\,dx = \frac{1}{7}\sqrt{(x^2-a^2)^7} + \frac{a^2}{5}\sqrt{(x^2-a^2)^5}$$

175. $$\int \frac{\sqrt{(x^2-a^2)^3}\,dx}{x} = \frac{1}{3}\sqrt{(x^2-a^2)^3} - a^2\sqrt{x^2-a^2} + a^3\,\mathrm{arccos}\,\frac{a}{x}$$

176. $$\int \frac{\sqrt{(x^2-a^2)^3}\,dx}{x^2}$$
$$= -\frac{1}{x}\sqrt{(x^2-a^2)^3} + \frac{3x}{2}\sqrt{x^2-a^2} - \frac{3a^2}{2}\log|x+\sqrt{x^2-a^2}|$$

177. $$\int \frac{\sqrt{(x^2-a^2)^3}\,dx}{x^3} = -\frac{\sqrt{(x^2-a^2)^3}}{2x^2} + \frac{3}{2}\sqrt{x^2-a^2} - \frac{3a}{2}\,\mathrm{arccos}\,\frac{a}{x}$$

178. $$\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$$

179. $$\int \frac{x\,dx}{\sqrt{(x^2-a^2)^3}} = -\frac{1}{\sqrt{x^2-a^2}}$$

180. $$\int \frac{x^2\,dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{\sqrt{x^2-a^2}} + \log|x+\sqrt{x^2-a^2}|$$

181. $$\int \frac{x^3\,dx}{\sqrt{(x^2-a^2)^3}} = \sqrt{x^2-a^2} - \frac{a^2}{\sqrt{x^2-a^2}}$$

182. $$\int \frac{dx}{x\sqrt{(x^2-a^2)^3}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\,\mathrm{arccos}\,\frac{a}{x}$$

183. $$\int \frac{dx}{x^2\sqrt{(x^2-a^2)^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{x^2-a^2}}{x} + \frac{x}{\sqrt{x^2-a^2}}\right)$$

184. $$\int \frac{dx}{x^3\sqrt{(x^2-a^2)^3}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}} - \frac{3}{2a^5}\,\mathrm{arccos}\,\frac{a}{x}$$

185. $$\int F(x, \sqrt{x^2-a^2})\,dx = a\int F(a\sec z, a\tan z)\sec z\tan z\,dz.$$

$(x = a\sec z)$

FORMS INVOLVING $\sqrt{ax^2+bx+c}$, WHERE $ax^2+bx+c>0$

Let $X = ax^2+bx+c$, $D = b^2-4ac$, $k = \frac{4a}{D}$, $l = an^2-bmn+cm^2$.

186. $$\int \sqrt{X}\,dx = \frac{2ax+b}{4a}\sqrt{X} - \frac{b^2-4ac}{8a}\int\frac{dx}{\sqrt{X}}$$

187. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3a} - \frac{b}{2a}\int\sqrt{X}\,dx$$

188. $$\int x^2\sqrt{X}\,dx = \left(x-\frac{5b}{6a}\right)\frac{X\sqrt{X}}{4a} + \frac{5b^2-4ac}{16a^2}\int\sqrt{X}\,dx$$

189. $$\int\frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int\frac{dx}{\sqrt{X}} + c\int\frac{dx}{x\sqrt{X}}$$

190. $$\int\frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int\frac{dx}{x\sqrt{X}} + a\int\frac{dx}{\sqrt{X}}$$

191. $$\int\frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{a}}\log\left|2ax+b+2\sqrt{a}\,\sqrt{X}\right| \qquad (a>0)$$

192. $$\int\frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-a}}\arcsin\left(\frac{-2ax-b}{\sqrt{D}}\right) \qquad (a<0)$$

193. $$\int\frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{X}}$$

194. $$\int\frac{x^2\,dx}{\sqrt{X}} = \left(\frac{2ax-3b}{4a^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8a^2}\int\frac{dx}{\sqrt{X}}$$

195. $$\int\frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{c}}\log\left|\frac{\sqrt{X}+\sqrt{c}}{x}+\frac{b}{2\sqrt{c}}\right| \qquad (c>0)$$

196. $$\int\frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-c}}\arcsin\left(\frac{bx+2c}{x\sqrt{D}}\right) \qquad (c<0)$$

197. $$\int\frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx} \qquad (c=0)$$

198. $$\int\frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int\frac{dx}{x\sqrt{X}}$$

199. $$\int(mx+n)\sqrt{X}\,dx = \frac{mX\sqrt{X}}{3a} + \frac{2an-bm}{2a}\int\sqrt{X}\,dx$$

200. $$\int\frac{\sqrt{X}\,dx}{mx+n} = \frac{\sqrt{X}}{m} + \frac{bm-2an}{2m^2}\int\frac{dx}{\sqrt{X}} + \frac{an^2-bmn+cm^2}{m^2}\int\frac{dx}{(mx+n)\sqrt{X}}$$

201. $$\int\frac{(mx+n)\,dx}{\sqrt{X}} = \frac{m\sqrt{X}}{a} + \frac{2an-bm}{2a}\int\frac{dx}{\sqrt{X}}$$

202. $$\int\frac{dx}{(mx+n)\sqrt{X}} = \frac{1}{\sqrt{l}}\log\left|\frac{2l+(bm-2an)(mx+n)-2m\sqrt{lX}}{mx+n}\right| \qquad (l>0)$$

203. $$\int\frac{dx}{(mx+n)\sqrt{X}} = \frac{1}{\sqrt{-l}}\arcsin\left[\frac{(bm-2an)(mx+n)+2l}{m(mx+n)\sqrt{D}}\right] \qquad (l<0)$$

204. $$\int\frac{dx}{(mx+n)\sqrt{X}} = -\frac{2m\sqrt{X}}{(bm-2an)(mx+n)} \qquad (l=0)$$

205. $$\int X\sqrt{X}\,dx = \frac{(2ax+b)\sqrt{X}}{8a}\left(X-\frac{3D}{8a}\right) + \frac{3D^2}{128a^2}\int\frac{dx}{\sqrt{X}}$$

206. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a}\int X\sqrt{X}\,dx$$

207. $$\int\frac{dx}{X\sqrt{X}} = -\frac{4ax+2b}{D\sqrt{X}} \qquad (D\neq 0)$$

208. $$\int\frac{dx}{X\sqrt{X}} = -\frac{1}{2\sqrt{a^3}\left(x+\frac{b}{2a}\right)^2} \qquad (D=0)$$

209. $$\int\frac{x\,dx}{X\sqrt{X}} = \frac{2bx+4c}{D\sqrt{X}}$$

210. $$\int X^m\sqrt{X}\,dx = \frac{(2ax+b)X^m\sqrt{X}}{4a(m+1)} - \frac{2m+1}{2(m+1)k}\int X^{m-1}\sqrt{X}\,dx$$

211. $$\int xX^m\sqrt{X}\,dx = \frac{X^{m+1}\sqrt{X}}{a(2m+3)} - \frac{b}{2a}\int X^m\sqrt{X}\,dx$$

212. $$\int\frac{X^m\sqrt{X}\,dx}{x} = \frac{X^m\sqrt{X}}{2m+1} + c\int\frac{X^{m-1}\sqrt{X}\,dx}{x} + \frac{b}{2}\int X^{m-1}\sqrt{X}\,dx$$

213. $$\int\frac{dx}{X^m\sqrt{X}} = -\frac{(4ax+2b)\sqrt{X}}{(2m-1)DX^m} - \frac{2k(m-1)}{2m-1}\int\frac{dx}{X^{m-1}\sqrt{X}}$$

214. $$\int \frac{x\,dx}{X^m\sqrt{X}} = -\frac{\sqrt{X}}{(2m-1)aX^m} - \frac{b}{2a}\int \frac{dx}{X^m\sqrt{X}}$$

215. $$\int \frac{dx}{xX^m\sqrt{X}} = \frac{\sqrt{X}}{(2m-1)cX^m} + \frac{1}{c}\int \frac{dx}{xX^{m-1}\sqrt{X}} - \frac{b}{2c}\int \frac{dx}{X^m\sqrt{X}}$$

216. $$\int \frac{x^m X^n\,dx}{\sqrt{X}} = \frac{x^{m-1}X^n\sqrt{X}}{(m+2n)a} - \frac{b(2m+2n-1)}{2a(m+2n)}\int \frac{x^{m-1}X^n\,dx}{\sqrt{X}}$$
$$- \frac{c(m-1)}{a(m+2n)}\int \frac{x^{m-2}X^n\,dx}{\sqrt{X}}$$

217. $$\int \frac{X^n\,dx}{x^m\sqrt{X}} = -\frac{X^{n-1}\sqrt{X}}{(m-1)x^{m-1}} + \frac{(2n-1)b}{2(m-1)}\int \frac{X^{n-1}\,dx}{x^{m-1}\sqrt{X}}$$
$$+ \frac{(2n-1)a}{m-1}\int \frac{X^{n-1}\,dx}{x^{m-2}\sqrt{X}} \qquad (m \neq 1)$$

218. $$\int \frac{x^m\,dx}{X^n\sqrt{X}} = \frac{1}{a}\int \frac{x^{m-2}\,dx}{X^{n-1}\sqrt{X}} - \frac{b}{a}\int \frac{x^{m-1}\,dx}{X^n\sqrt{X}} - \frac{c}{a}\int \frac{x^{m-2}\,dx}{X^n\sqrt{X}}$$

219. $$\int \frac{dx}{x^m X^n\sqrt{X}} = -\frac{\sqrt{X}}{(m-1)cx^{m-1}X^n}$$
$$- \frac{(2m+2n-3)b}{2c(m-1)}\int \frac{dx}{x^{m-1}X^n\sqrt{X}}$$
$$- \frac{(m+2n-2)a}{(m-1)c}\int \frac{dx}{x^{m-2}X^n\sqrt{X}}$$

FORMS INVOLVING $\sqrt{2ax - x^2}$, WHERE $2ax - x^2 > 0$

220. $$\int \sqrt{2ax - x^2}\,dx = \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x-a}{a}\right)$$

221. $$\int x\sqrt{2ax - x^2}\,dx = -\frac{3a^2 + ax - 2x^2}{6}\sqrt{2ax - x^2} + \frac{a^3}{2}\arcsin\left(\frac{x-a}{a}\right)$$

222. $$\int x^m\sqrt{2ax - x^2}\,dx$$
$$= -\frac{x^{m-1}\sqrt{(2ax - x^2)^3}}{m+2} + \frac{a(2m+1)}{m+2}\int x^{m-1}\sqrt{2ax - x^2}\,dx$$

223. $$\int \frac{\sqrt{2ax - x^2}\,dx}{x} = \sqrt{2ax - x^2} + a\arcsin\left(\frac{x-a}{a}\right)$$

224. $$\int \frac{\sqrt{2ax - x^2}\,dx}{x^m} = -\frac{\sqrt{(2ax - x^2)^3}}{a(2m-3)x^m} + \frac{m-3}{a(2m-3)}\int \frac{\sqrt{2ax - x^2}\,dx}{x^{m-1}}$$

225. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \arcsin\left(\frac{x-a}{a}\right)$$

226. $$\int \frac{x\,dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a\arcsin\left(\frac{x-a}{a}\right)$$

227. $$\int \frac{x^m\,dx}{\sqrt{2ax - x^2}} = -\frac{x^{m-1}\sqrt{2ax - x^2}}{m} + \frac{a(2m-1)}{m}\int \frac{x^{m-1}\,dx}{\sqrt{2ax - x^2}}$$

228. $$\int \frac{dx}{x\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax}$$

229. $$\int \frac{dx}{x^m\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{a(2m-1)x^m} + \frac{m-1}{a(2m-1)}\int \frac{dx}{x^{m-1}\sqrt{2ax - x^2}}$$

MISCELLANEOUS ALGEBRAIC FORMS

230. $$\int \sqrt{\frac{a+x}{b+x}}\,dx = \sqrt{(a+x)(b+x)} + (a-b)\log\left(\sqrt{a+x} + \sqrt{b+x}\right)$$
$$(a + x > 0 \text{ and } b + x > 0)$$

231. $$\int \sqrt{\frac{a+x}{b-x}}\,dx = -\sqrt{(a+x)(b-x)} - (a+b)\arcsin\sqrt{\frac{b-x}{a+b}}$$
$$(a + x \text{ and } b - x \text{ have the same sign})$$

232. $$\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b)\arcsin\sqrt{\frac{b+x}{a+b}}$$
$$(a - x \text{ and } b + x \text{ have the same sign})$$

233. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = -\sqrt{1-x^2} + \arcsin x$$
$$(1 + x \text{ and } 1 - x \text{ have the same sign})$$

234. $$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}}$$
$$(x - a \text{ and } b - x \text{ have the same sign})$$

235. $$\int \frac{dx}{ax^3 + b} = \frac{k}{3b}\left[\sqrt{3}\arctan\frac{2x-k}{k\sqrt{3}} + \log\left|\frac{k+x}{\sqrt{x^2 - kx + k^2}}\right|\right]$$
$$\left(b \neq 0,\ k = \sqrt[3]{\frac{b}{a}}\right)$$

236. $\displaystyle\int \frac{x\,dx}{ax^3+b} = \frac{1}{3ak}\left[\sqrt{3}\arctan\frac{2x-k}{k\sqrt{3}} - \log\left|\frac{k+x}{\sqrt{x^2-kx+k^2}}\right|\right]$

$\left(b \neq 0,\ k = \sqrt[3]{\frac{a}{b}}\right)$

237. $\displaystyle\int \frac{dx}{x(ax^m+b)} = \frac{1}{bm}\log\left|\frac{x^m}{ax^m+b}\right|$ $(b \neq 0)$

238. $\displaystyle\int \frac{dx}{\sqrt{(2ax-x^2)^3}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}$

239. $\displaystyle\int \frac{x\,dx}{\sqrt{(2ax-x^2)^3}} = \frac{x}{a\sqrt{2ax-x^2}}$

240. $\displaystyle\int \frac{dx}{\sqrt{2ax+x^2}} = \log|x+a+\sqrt{2ax+x^2}|$

241. $\displaystyle\int F(x^{\frac{1}{n}})\,dx = n\int F(z)z^{n-1}\,dz$ $(x = z^n)$

242. $\displaystyle\int F[x, (ax+b)^{\frac{1}{n}}]\,dx = \frac{n}{a}\int F\left(\frac{z^n-b}{a}, z\right)z^{n-1}\,dz$

$(ax+b = z^n)$

BINOMIAL REDUCTION FORMULAS

In the following formulas m, n, and p may be any numbers for which the denominators do not become zero, and $u = ax^n + b$.

243. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{x^{m+1}u^p}{m+np+1} + \frac{bnp}{m+np+1}\int x^m u^{p-1}\,dx$

244. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{x^{m-n+1}u^{p+1}}{a(m+np+1)} - \frac{b(m-n+1)}{a(m+np+1)}\int x^{m-n}u^p\,dx$

245. $\displaystyle\int \frac{(ax^n+b)^p\,dx}{x^m} = \frac{u^p}{(np-m+1)x^{m-1}} + \frac{bnp}{np-m+1}\int\frac{u^{p-1}\,dx}{x^m}$

246. $\displaystyle\int \frac{(ax^n+b)^p\,dx}{x^m} = -\frac{u^{p+1}}{b(m-1)x^{m-1}} - \frac{a(m-n-np-1)}{b(m-1)}\int\frac{u^p\,dx}{x^{m-n}}$

247. $\displaystyle\int \frac{x^m\,dx}{(ax^n+b)^p} = \frac{x^{m+1}}{bn(p-1)u^{p-1}} - \frac{m+n-np+1}{bn(p-1)}\int\frac{x^m\,dx}{u^{p-1}}$

248. $\displaystyle\int \frac{x^m\,dx}{(ax^n+b)^p} = \frac{x^{m-n+1}}{a(m-np+1)u^{p-1}} - \frac{b(m-n+1)}{a(m-np+1)}\int\frac{x^{m-n}\,dx}{u^p}$

249. $\displaystyle\int \frac{dx}{x^m(ax^n+b)^p} = \frac{1}{bn(p-1)x^{m-1}u^{p-1}} + \frac{m-n+np-1}{bn(p-1)}\int\frac{dx}{x^m u^{p-1}}$

250. $\displaystyle\int \frac{dx}{x^m(ax^n+b)^p} = -\frac{1}{b(m-1)x^{m-1}u^{p-1}} - \frac{a(m-n+np-1)}{b(m-1)}\int\frac{dx}{x^{m-n}u^p}$

FORMS INVOLVING SIN X

When the result contains an infinite series, the interval of convergence is given in parentheses.

251. $\displaystyle\int \sin x\,dx = -\cos x$

252. $\displaystyle\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4}$

253. $\displaystyle\int \sin^3 x\,dx = \frac{\cos^3 x}{3} - \cos x$

254. $\displaystyle\int \sin^4 x\,dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$

255. $\displaystyle\int \sin^{2m} x\,dx = -\frac{\sin^{2m-1}x\cos x}{2m} + \frac{2m-1}{2m}\int\sin^{2(m-1)}x\,dx$

256. $\displaystyle\int \sin^{2m+1} x\,dx = \int(1-\cos^2 x)^m \sin x\,dx$

Expand and use Formula 316.

257. $\displaystyle\int \frac{dx}{\sin x} = \int \csc x\,dx = \log\left|\tan\frac{x}{2}\right| = \log|\csc x - \operatorname{ctn} x|$

258. $\displaystyle\int \frac{dx}{\sin^2 x} = \int \csc^2 x\,dx = -\operatorname{ctn} x$

259. $\displaystyle\int \frac{dx}{\sin^m x} = \int \csc^m x\,dx = -\frac{\cos x}{(m-1)\sin^{m-1}x} + \frac{m-2}{m-1}\int\frac{dx}{\sin^{m-2}x}$

$(m \neq 1)$

260. $\displaystyle\int x\sin x\,dx = \sin x - x\cos x$

261. $\displaystyle\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x$

262. $\displaystyle\int x^m \sin x\,dx = -x^m\cos x + mx^{m-1}\sin x$

$\displaystyle - m(m-1)\int x^{m-2}\sin x\,dx$

263. $\displaystyle\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8}$

264. $$\int x^2 \sin^2 x\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4}$$

265. $$\int \frac{x\,dx}{\sin x} = x + \frac{x^3}{3\cdot 3!} + \frac{7x^5}{3\cdot 5\cdot 5!} + \frac{31x^7}{3\cdot 7\cdot 7!} + \frac{127x^9}{3\cdot 5\cdot 9!} + \cdots \qquad (x^2 < \pi^2)$$

266. $$\int \frac{x^2\,dx}{\sin x} = \frac{x^2}{2} + \frac{x^4}{4\cdot 3!} + \frac{7x^6}{3\cdot 6\cdot 5!} + \frac{31x^8}{3\cdot 8\cdot 7!} + \frac{127x^{10}}{5\cdot 5\cdot 6\cdot 8!} + \cdots \qquad (x^2 < \pi^2)$$

267. $$\int \frac{x\,dx}{\sin^2 x} = -x\operatorname{ctn} x + \log|\sin x|$$

268. $$\int \frac{\sin x\,dx}{x} = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots \qquad (x^2 < \infty)$$

269. $$\int \frac{\sin x\,dx}{x^2} = -\frac{\sin x}{x} + \log|x| - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \cdots \qquad (0 < x^2 < \infty)$$

270. $$\int \frac{\sin x\,dx}{x^m} = -\frac{\sin x}{(m-1)x^{m-1}} - \frac{\cos x}{(m-1)(m-2)x^{m-2}} - \frac{1}{(m-1)(m-2)}\int \frac{\sin x\,dx}{x^{m-2}} \qquad (m \neq 1, 2)$$

271. $$\int x^m \sin^n x\,dx = \frac{1}{n^2}x^{m-1}\sin^{n-1} x(m\sin x - nx\cos x) + \frac{n-1}{n}\int x^m \sin^{n-2} x\,dx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin^n x\,dx$$

272. $$\int \frac{x^m\,dx}{\sin^n x\,dx} = -\frac{x^{m-1}[m\sin x + (n-2)x\cos x]}{(n-1)(n-2)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{x^m\,dx}{\sin^{n-2} x} + \frac{m(m-1)}{(n-1)(n-2)}\int \frac{x^{m-2}\,dx}{\sin^{n-2} x} \qquad (n \neq 1, 2)$$

273. $$\int \frac{\sin^n x\,dx}{x^m} = -\frac{\sin^{n-1} x[(m-2)\sin x + nx\cos x]}{(m-1)(m-2)x^{m-1}} - \frac{n^2}{(m-1)(m-2)}\int \frac{\sin^n x\,dx}{x^{m-2}} + \frac{n(n-1)}{(m-1)(m-2)}\int \frac{\sin^{n-2} x\,dx}{x^{m-2}} \qquad (m \neq 1, 2)$$

274. $$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad (a \neq b)$$

275. $$\int \frac{dx}{1+\sin x} = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \tan x - \sec x$$

276. $$\int \frac{dx}{1-\sin x} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tan x + \sec x$$

277. $$\int \frac{dx}{a + b\sin x} = \frac{-2}{\sqrt{a^2-b^2}}\arctan\left[\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] \qquad (a^2 > b^2)$$

278. $$\int \frac{dx}{a + b\sin x} = \frac{-1}{\sqrt{b^2-a^2}}\log\frac{b + a\sin x + \sqrt{b^2-a^2}\cos x}{a + b\sin x} \qquad (a^2 < b^2)$$

279. $$\int F(\sin x)\,dx = \int F(z)\frac{dz}{\sqrt{1-z^2}} \qquad (z = \sin x)$$

280. $$\int F(\sin x)\,dx = 2\int F\left(\frac{2z}{1+z^2}\right)\frac{dz}{1+z^2} \qquad \left(z = \tan\frac{x}{2}\right)$$

FORMS INVOLVING COS x

When the result contains an infinite series, the interval of convergence is given in parentheses.

281. $$\int \cos x\,dx = \sin x$$

282. $$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

283. $$\int \cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3}$$

284. $$\int \cos^4 x\,dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$$

285. $$\int \cos^{2m} x\,dx = \frac{1}{2m}\cos^{2m-1} x\sin x + \frac{2m-1}{2m}\int \cos^{2(m-1)} x\,dx$$

286. $$\int \cos^{2m+1} x\,dx = \int (1 - \sin^2 x)^m \cos x\,dx.$$

Expand and use Formula 315.

287. $$\int \frac{dx}{\cos x} = \int \sec x\,dx = \log|\sec x + \tan x| = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|$$

288. $$\int \frac{dx}{\cos^2 x} = \int \sec^2 x\,dx = \tan x$$

289. $$\int \frac{dx}{\cos^m x} = \int \sec^m x\,dx = \frac{\sin x}{(m-1)\cos^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\cos^{m-2} x} \qquad (m \neq 1)$$

290. $$\int x\cos x\,dx = \cos x + x\sin x$$

291. $$\int x^2 \cos x\,dx = 2x\cos x + (x^2 - 2)\sin x$$

292. $$\int x^m \cos x\,dx = x^m \sin x + mx^{m-1}\cos x - m(m-1)\int x^{m-2}\cos x\,dx$$

293. $$\int x\cos^2 x\,dx = \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$$

294. $$\int x^2\cos^2 x\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x + \frac{x\cos 2x}{4}$$

295. $$\int \frac{x\,dx}{\cos x} = \frac{x^2}{2} + \frac{x^4}{4\cdot 2!} + \frac{5x^6}{6\cdot 4!} + \frac{61x^8}{8\cdot 6!} + \frac{1385x^{10}}{10\cdot 8!} + \cdots \qquad \left(x^2 < \frac{\pi^2}{4}\right)$$

296. $$\int \frac{x^2\,dx}{\cos x} = \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{5x^7}{7\cdot 4!} + \frac{61x^9}{9\cdot 6!} + \frac{1385x^{11}}{11\cdot 8!} + \cdots \qquad \left(x^2 < \frac{\pi^2}{4}\right)$$

297. $$\int \frac{x\,dx}{\cos^2 x} = x\tan x + \log|\cos x|$$

298. $$\int \frac{\cos x\,dx}{x} = \log|x| - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \cdots \qquad (0 < x^2 < \infty)$$

299. $$\int \frac{\cos x\,dx}{x^2} = -\frac{\cos x}{x} - x + \frac{x^3}{3\cdot 3!} - \frac{x^5}{5\cdot 5!} + \frac{x^7}{7\cdot 7!} - \cdots \qquad (0 < x^2 < \infty)$$

300. $$\int \frac{\cos x\,dx}{x^m} = -\frac{\cos x}{(m-1)x^{m-1}} + \frac{\sin x}{(m-1)(m-2)x^{m-2}} - \frac{1}{(m-1)(m-2)}\int \frac{\cos x\,dx}{x^{m-2}}$$

301. $$\int x^m \cos^n x\,dx = \frac{1}{n^2}x^{m-1}\cos^{n-1} x(m\cos x + nx\sin x) + \frac{n-1}{n}\int x^m \cos^{n-2} x\,dx - \frac{m(m-1)}{n^2}\int x^{m-2}\cos^n x\,dx$$

302. $$\int \frac{x^m\,dx}{\cos^n x} = -\frac{x^{m-1}[m\cos x - (n-2)x\sin x]}{(n-1)(n-2)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{x^m\,dx}{\cos^{n-2} x} + \frac{m(m-1)}{(n-1)(n-2)}\int \frac{x^{m-2}\,dx}{\cos^{n-2} x} \qquad (n \neq 1, 2)$$

303. $$\int \frac{\cos^n x\,dx}{x^m} = \frac{\cos^{n-1} x[nx\sin x - (m-2)\cos x]}{(m-1)(m-2)x^{m-1}} - \frac{n^2}{(m-1)(m-2)}\int \frac{\cos^n x\,dx}{x^{m-2}} + \frac{n(n-1)}{(m-1)(m-2)}\int \frac{\cos^{n-2} x\,dx}{x^{m-2}} \qquad (m \neq 1, 2)$$

304. $$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad (a \neq b)$$

305. $$\int \frac{dx}{1+\cos x} = \tan\frac{x}{2} = \csc x - \operatorname{ctn} x$$

306. $$\int \frac{dx}{1-\cos x} = -\operatorname{ctn}\frac{x}{2} = -\csc x - \operatorname{ctn} x$$

307. $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right] \qquad (a^2 > b^2)$$

308. $$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{b + a\cos x + \sqrt{b^2-a^2}\sin x}{a+b\cos x}\right| \qquad (a^2 < b^2)$$

309. $$\int F(\cos x)\,dx = -\int F(z)\frac{dz}{\sqrt{1-z^2}} \qquad (z = \cos x)$$

310. $$\int F(\cos x)\,dx = 2\int F\left(\frac{1-z^2}{1+z^2}\right)\frac{dz}{1+z^2} \qquad \left(z = \tan\frac{x}{2}\right)$$

FORMS INVOLVING BOTH SIN x AND COS x

311. $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x$

312. $\int \sin ax \cos bx \, dx = -\frac{\cos (a-b)x}{2(a-b)} - \frac{\cos (a+b)x}{2(a+b)} \qquad (a^2 \neq b^2)$

313. $\int \frac{\sin x \, dx}{\cos x} = \int \tan x \, dx = \log |\sec x|$

314. $\int \frac{\cos x \, dx}{\sin x} = \int \operatorname{ctn} x \, dx = \log |\sin x|$

315. $\int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1} \qquad (m \neq -1)$

316. $\int \sin x \cos^m x \, dx = -\frac{\cos^{m+1} x}{m+1} \qquad (m \neq -1)$

317. $\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{\sin 4x}{32}$

318. $\int \frac{\sin^2 x \, dx}{\cos x} = -\sin x + \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|$

319. $\int \frac{\sin x \, dx}{\cos^2 x} = \sec x$

320. $\int \frac{\cos^2 x \, dx}{\sin x} = \cos x + \log \left| \tan \frac{x}{2} \right|$

321. $\int \frac{\cos x \, dx}{\sin^2 x} = -\csc x$

322. $\int \frac{dx}{\sin x \cos x} = \log |\tan x|$

323. $\int \frac{dx}{\sin^2 x \cos x} = -\csc x + \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|$

324. $\int \frac{dx}{\sin x \cos^2 x} = \sec x + \log \left| \tan \frac{x}{2} \right|$

325. $\int \frac{dx}{\sin^m x \cos x} = -\frac{1}{(m-1) \sin^{m-1} x} + \int \frac{dx}{\sin^{m-2} x \cos x} \qquad (m \neq 1)$

326. $\int \frac{dx}{\sin x \cos^m x} = \frac{1}{(m-1) \cos^{m-1} x} + \int \frac{dx}{\sin x \cos^{m-2} x} \qquad (m \neq 1)$

327. $\int \frac{dx}{\sin^2 x \cos^2 x} = \tan x - \operatorname{ctn} x$

328. $\int \sin^m x \cos^n x \, dx$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx \qquad (m \neq -n)$$

329. $\int \sin^m x \cos^n x \, dx$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx \qquad (m \neq -n)$$

330. $\int \frac{\sin^m x \, dx}{\cos^n x} = \frac{\sin^{m+1} x}{(n-1) \cos^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\sin^m x \, dx}{\cos^{n-2} x} \qquad (n \neq 1)$

331. $\int \frac{\sin^m x \, dx}{\cos^n x} = -\frac{\sin^{m-1} x}{(m-n) \cos^{n-1} x} + \frac{m-1}{m-n} \int \frac{\sin^{m-2} x \, dx}{\cos^n x} \qquad (m \neq n)$

332. $\int \frac{dx}{\sin^m x \cos^n x}$

$$= -\frac{1}{(m-1) \sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cos^n x} \qquad (m \neq 1)$$

333. $\int \frac{dx}{\sin^m x \cos^n x}$

$$= \frac{1}{(n-1) \sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cos^{n-2} x} \qquad (n \neq 1)$$

334. $\int \frac{\cos^n x \, dx}{\sin^m x} = -\frac{\cos^{n+1} x}{(m-1) \sin^{m-1} x} - \frac{n-m+2}{m-1} \int \frac{\cos^n x \, dx}{\sin^{m-2} x} \qquad (m \neq 1)$

335. $\int \frac{\cos^n x \, dx}{\sin^m x} = \frac{\cos^{n-1} x}{(n-m) \sin^{m-1} x} + \frac{n-1}{n-m} \int \frac{\cos^{n-2} x \, dx}{\sin^m x} \qquad (m \neq n)$

336. $\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right|$

337. $\int \frac{dx}{\sin x - \cos x} = \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{8} \right) \right|$

338. $$\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \log \left| \tan \frac{1}{2}\left(x + \arctan \frac{b}{a}\right) \right|$$

339. $$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{1}{ab} \arctan \left(\frac{a}{b} \tan x\right)$$

340. $$\int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x} = \frac{1}{2ab} \log \left| \frac{a \sin x - b \cos x}{a \sin x + b \cos x} \right|$$

341. $$\int \frac{dx}{a + b \sin x + c \cos x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \arctan \frac{b + (a - c) \tan \frac{x}{2}}{\sqrt{a^2 - b^2 - c^2}} \qquad (a^2 > b^2 + c^2)$$

342. $$\int \frac{dx}{a + b \sin x + c \cos x} = \frac{1}{\sqrt{b^2 + c^2 - a^2}} \log \left| \frac{b - \sqrt{b^2 + c^2 - a^2} + (a - c) \tan \frac{x}{2}}{b + \sqrt{b^2 + c^2 - a^2} + (a - c) \tan \frac{x}{2}} \right| \qquad (a^2 < b^2 + c^2)$$

343. $$\int F(\sin x, \cos x)\, dx = \int F(z, \sqrt{1 - z^2}) \frac{dz}{\sqrt{1 - z^2}} \qquad (z = \sin x)$$

344. $$\int F(\sin x, \cos x)\, dx = 2 \int F\left(\frac{2z}{1 + z^2}, \frac{1 - z^2}{1 + z^2}\right) \frac{dz}{1 + z^2} \qquad \left(z = \tan \frac{x}{2}\right)$$

FORMS INVOLVING TAN x, CTN x, SEC x, CSC x

345. $$\int \tan x\, dx = \log |\sec x|$$

346. $$\int \tan^2 x\, dx = \tan x - x$$

347. $$\int \tan^m x\, dx = \frac{\tan^{m-1} x}{m - 1} - \int \tan^{m-2} x\, dx \qquad (m \neq 1)$$

348. $$\int \operatorname{ctn} x\, dx = \log |\sin x|$$

349. $$\int \operatorname{ctn}^2 x\, dx = -\operatorname{ctn} x - x$$

350. $$\int \operatorname{ctn}^m x\, dx = -\frac{\operatorname{ctn}^{m-1} x}{m - 1} - \int \operatorname{ctn}^{m-2} x\, dx \qquad (m \neq 1)$$

351. $$\int \sec x\, dx = \log |\sec x + \tan x| = \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) \right|$$

352. $$\int \sec^2 x\, dx = \tan x$$

353. $$\int \sec^m x\, dx = \frac{\sin x}{(m - 1) \cos^{m-1} x} + \frac{m - 2}{m - 1} \int \sec^{m-2} x\, dx \qquad (m \neq 1)$$

354. $$\int \csc x\, dx = \log |\csc x - \operatorname{ctn} x| = \log \left| \tan \frac{x}{2} \right|$$

355. $$\int \csc^2 x\, dx = -\operatorname{ctn} x$$

356. $$\int \csc^m x\, dx = -\frac{\cos x}{(m - 1) \sin^{m-1} x} + \frac{m - 2}{m - 1} \int \csc^{m-2} x\, dx \qquad (m \neq 1)$$

357. $$\int \tan x \sec x\, dx = \sec x$$

358. $$\int \tan x \sec^m x\, dx = \frac{\sec^m x}{m} \qquad (m \neq 0)$$

359. $$\int \tan^m x \sec^2 x\, dx = \frac{\tan^{m+1} x}{m + 1} \qquad (m \neq -1)$$

360. $$\int \frac{\sec^2 x\, dx}{\tan x} = \log |\tan x|$$

361. $$\int \operatorname{ctn} x \csc x\, dx = -\csc x$$

362. $$\int \operatorname{ctn} x \csc^m x\, dx = -\frac{\csc^m x}{m} \qquad (m \neq 0)$$

363. $$\int \operatorname{ctn}^m x \csc^2 x\, dx = -\frac{\operatorname{ctn}^{m+1} x}{m + 1} \qquad (m \neq -1)$$

364. $$\int \frac{\csc^2 x\, dx}{\operatorname{ctn} x} = \log |\tan x|$$

365. $\int \frac{dx}{a + b \tan x} = \frac{1}{a^2 + b^2} [ax + b \log |a \cos x + b \sin x|]$

366. $\int \frac{dx}{a + b \operatorname{ctn} x} = \frac{1}{a^2 + b^2} [ax - b \log |a \sin x + b \cos x|]$

EXPONENTIAL FORMS

For integrals involving a^x, substitute $a^x = e^{(\log a) x}$ and use the forms that follow.

367. $\int e^{ax}\, dx = \frac{1}{a} e^{ax}$

368. $\int x e^{ax}\, dx = \frac{e^{ax}}{a^2} (ax - 1)$

369. $\int x^m e^{ax}\, dx = \frac{1}{a} x^m e^{ax} - \frac{m}{a} \int x^{m-1} e^{ax}\, dx$

370. $\int x^p e^{ax}\, dx$

$$= \frac{e^{ax}}{a^{p+1}} [(ax)^p - p(ax)^{p-1} + p(p-1)(ax)^{p-2} - \cdots + (-1)^p p!]$$
$(p = \text{integer})$

371. $\int \frac{e^{ax}\, dx}{x} = \log |x| + ax + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \cdots$ $(0 < x^2 < \infty)$

372. $\int \frac{e^{ax}\, dx}{x^m} = -\frac{e^{ax}}{(m-1)x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}\, dx}{x^{m-1}}$ $(m \neq 1)$

373. $\int \frac{e^{ax}\, dx}{b + ce^{ax}} = \frac{1}{ac} \log |b + ce^{ax}|$

374. $\int \frac{dx}{b + ce^{ax}} = \frac{1}{ab} \log \left| \frac{e^{ax}}{b + ce^{ax}} \right|$

375. $\int \frac{dx}{ae^{cx} + be^{-cx}} = \frac{1}{c\sqrt{ab}} \arctan \left(e^{cx} \sqrt{\frac{a}{b}} \right)$ $(ab > 0)$

376. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$

377. $\int e^{ax} \sin^m bx\, dx = \frac{e^{ax}(a \sin bx - mb \cos bx) \sin^{m-1} bx}{a^2 + m^2 b^2}$

$$+ \frac{m(m-1)b^2}{a^2 + m^2 b^2} \int e^{ax} \sin^{m-2} bx\, dx$$

378. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

379. $\int e^{ax} \cos^m bx\, dx = \frac{e^{ax}(a \cos bx + mb \sin bx) \cos^{m-1} bx}{a^2 + m^2 b^2}$

$$+ \frac{m(m-1)b^2}{a^2 + m^2 b^2} \int e^{ax} \cos^{m-2} bx\, dx$$

380. $\int e^{ax} \log bx\, dx = \frac{1}{a} e^{ax} \log bx - \frac{1}{a} \int \frac{e^{ax}\, dx}{x}$ $(bx > 0)$

LOGARITHMIC FORMS

In these forms, $x > 0$.

381. $\int \log x\, dx = x(\log x - 1)$

382. $\int (\log x)^m\, dx = x(\log x)^m - m \int (\log x)^{m-1}\, dx$

383. $\int \frac{dx}{\log x} = \log |\log x| + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \cdots$ $(0 < x < \infty)$

384. $\int \frac{dx}{(\log x)^m} = -\frac{x}{(m-1)(\log x)^{m-1}} + \frac{1}{m-1} \int \frac{dx}{(\log x)^{m-1}}$

385. $\int x^m \log x\, dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right]$ $(m \neq -1)$

386. $\int \frac{(\log x)^m\, dx}{x} = \frac{(\log x)^{m+1}}{m+1}$ $(m \neq -1)$

387. $\int \frac{dx}{x \log x} = \log |\log x|$

388. $\int x^m (\log x)^n\, dx = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1}\, dx$ $(m, n \neq -1)$

389. $\int \frac{x^m\, dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m\, dx}{(\log x)^{n-1}}$ $(n \neq 1)$

390. $\int \frac{x^m\, dx}{\log x} = \log |\log x| + (m+1) \log x + \frac{(m+1)^2 (\log x)^2}{2 \cdot 2!}$

$$+ \frac{(m+1)^3 (\log x)^3}{3 \cdot 3!} + \cdots$$
$(0 < x < \infty)$

391. $\int \sin(\log x)\,dx = \frac{x}{2}[\sin(\log x) - \cos(\log x)]$

392. $\int \cos(\log x)\,dx = \frac{x}{2}[\sin(\log x) + \cos(\log x)]$

FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

393. $\int \arcsin x\,dx = x \arcsin x + \sqrt{1-x^2}$

394. $\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

395. $\int x \arcsin x\,dx = \frac{1}{4}[(2x^2-1)\arcsin x + x\sqrt{1-x^2}]$

396. $\int \frac{\arcsin x\,dx}{x} = x + \frac{x^3}{2\cdot 3\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} + \cdots$ $(x^2 < 1)$

397. $\int x^m \arcsin x\,dx = \frac{x^{m+1}}{m+1}\arcsin x - \frac{1}{m+1}\int \frac{x^{m+1}\,dx}{\sqrt{1-x^2}}$ $(m \neq -1)$

398. $\int \arccos x\,dx = x \arccos x - \sqrt{1-x^2}$

399. $\int (\arccos x)^2\,dx = x(\arccos x)^2 - 2x - 2\sqrt{1-x^2}\arccos x$

400. $\int x \arccos x\,dx = \frac{1}{4}[(2x^2-1)\arccos x - x\sqrt{1-x^2}]$

401. $\int \frac{\arccos x\,dx}{x}$

$$= \frac{\pi}{2}\log|x| - x - \frac{x^3}{2\cdot 3\cdot 3} - \frac{1\cdot 3x^5}{2\cdot 4\cdot 5\cdot 5} - \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7\cdot 7} - \cdots$$ $(x^2 < 1)$

402. $\int x^m \arccos x\,dx = \frac{x^{m+1}}{m+1}\arccos x + \frac{1}{m+1}\int \frac{x^{m+1}\,dx}{\sqrt{1-x^2}}$ $(m \neq -1)$

403. $\int \arctan x\,dx = x \arctan x - \log\sqrt{1+x^2}$

404. $\int x^m \arctan x\,dx = \frac{x^{m+1}\arctan x}{m+1} - \frac{1}{m+1}\int \frac{x^{m+1}\,dx}{1+x^2}$ $(m \neq -1)$

405. $\int \operatorname{arcctn} x\,dx = x \operatorname{arcctn} x + \log\sqrt{1+x^2}$

406. $\int x^m \operatorname{arcctn} x\,dx = \frac{x^{m+1}}{m+1}\operatorname{arcctn} x + \frac{1}{m+1}\int \frac{x^{m+1}\,dx}{1+x^2}$ $(m \neq -1)$

407. $\int \operatorname{arcsec} x\,dx = x \operatorname{arcsec} x - \log|x + \sqrt{x^2-1}|$

408. $\int x^m \operatorname{arcsec} x\,dx = \frac{x^{m+1}}{m+1}\operatorname{arcsec} x - \frac{1}{m+1}\int \frac{x^m\,dx}{\sqrt{x^2-1}}$ $(m \neq -1)$

409. $\int \operatorname{arccsc} x\,dx = x \operatorname{arccsc} x + \log|x + \sqrt{x^2-1}|$

410. $\int x^m \operatorname{arccsc} x\,dx = \frac{x^{m+1}}{m+1}\operatorname{arccsc} x + \frac{1}{m+1}\int \frac{x^m\,dx}{\sqrt{x^2-1}}$ $(m \neq -1)$

FORMS INVOLVING HYPERBOLIC FUNCTIONS

411. $\int \sinh x\,dx = \cosh x$

412. $\int \sinh^2 x\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}$

413. $\int x \sinh x\,dx = x \cosh x - \sinh x$

414. $\int \cosh x\,dx = \sinh x$

415. $\int \cosh^2 x\,dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

416. $\int x \cosh x\,dx = x \sinh x - \cosh x$

417. $\int \tanh x\,dx = \log(\cosh x)$

418. $\int \tanh^2 x\,dx = x - \tanh x$

419. $\int \operatorname{ctnh} x\,dx = \log|\sinh x|$

420. $\int \operatorname{ctnh}^2 x\,dx = x - \operatorname{ctnh} x$

421. $\int \operatorname{sech} x\,dx = \arctan(\sinh x)$

422. $\int \operatorname{sech}^2 x\,dx = \tanh x$

423. $\int \operatorname{csch} x\,dx = \log\left|\tanh\frac{x}{2}\right|$

424. $\int \operatorname{csch}^2 x\,dx = -\operatorname{ctnh} x$

425. $\int \sinh x \cosh x\,dx = \frac{1}{4}\cosh 2x$

426. $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x$

427. $\int \operatorname{csch} x \operatorname{ctnh} x\,dx = -\operatorname{csch} x$

428. $\int \sinh ax \sinh bx\,dx = \frac{\sinh(a+b)x}{2(a+b)} - \frac{\sinh(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$

429. $\int \cosh ax \cosh bx\,dx = \frac{\sinh(a+b)x}{2(a+b)} + \frac{\sinh(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$

430. $\int \sinh ax \cosh bx\,dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$

14 Definite Integrals

In the following formulas, a, b, m, and n represent any positive real numbers; p and q represent positive integers; k represents any positive or negative real number.

THE GAMMA FUNCTION

431. $\int_0^\infty x^{m-1}e^{-x}\,dx = \Gamma(m)$

$$\Gamma(m+1) = m\Gamma(m) \qquad \Gamma(p) = (p-1)!$$

$$\Gamma(2) = \Gamma(1) = 1 \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(p+\tfrac{1}{2}) = \frac{1\cdot 3\cdot 5\cdots(2p-1)}{2^p}\sqrt{\pi}$$

432. $\int_0^\infty x^{m-1}e^{-ax}\,dx = \frac{1}{a^m}\Gamma(m)$

433. $\int_0^\infty e^{-x^m}\,dx = \Gamma\left(\frac{1}{m}+1\right)$

434. $\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1}dx = \frac{1}{m^n}\Gamma(n)$

435. $\int_0^1 (\log x)^{n-1}\,dx = (-1)^{n-1}\Gamma(n)$

THE BETA FUNCTION

436. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m, n)$

$$B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$B(p, n) = \frac{(p-1)!}{n(n+1)\cdots(n+p-1)}$$

$$B(p, q) = \frac{(p-1)!\,(q-1)!}{(p+q-1)!}$$

437. $\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = B(m, n)$

438. $\int_0^a x^{m-1}(a-x)^{n-1}\,dx = a^{m+n-1}B(m, n)$

439. $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}\,dx}{(a+x)^{m+n}} = \frac{B(m, n)}{a^n(a+1)^m}$

440. $\int_b^a (a-x)^{m-1}(x-b)^{n-1}\,dx = (a-b)^{m+n-1}B(m, n) \qquad (a > b)$

MISCELLANEOUS FORMS

441. $\int_1^\infty \frac{dx}{x^{m+1}} = \frac{1}{m}$

442. $\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n} = \frac{\pi}{n}\csc\frac{m\pi}{n} \qquad (m < n)$

443. $\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}$

444. $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$

445. $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

446. $\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$

447. $\int_0^\infty \frac{\sin ax}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos ax}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2a}}$

448. $\int_0^\infty \frac{\cos bx}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-ab}$

449. $\int_0^\infty \frac{x\sin bx}{a^2+x^2}\,dx = \frac{\pi}{2}e^{-ab}$

450. $\int_0^\infty \frac{\sin bx}{x(a^2+x^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-ab})$

451. $\int_0^\pi \sin^2 px\,dx = \int_0^\pi \cos^2 px\,dx = \frac{\pi}{2}$

452. $\int_0^\infty \frac{\sin^2 kx}{x^2}\,dx = \frac{\pi}{2}\cdot|k|$

453. $\int_0^{\frac{\pi}{p}} \sin px\cos px\,dx = \int_0^\pi \sin px\cos px\,dx = 0$

454. $\int_0^\pi \sin px\sin qx\,dx = \int_0^\pi \cos px\cos qx\,dx = 0 \qquad (p \neq q)$

455. $\int_0^\pi \sin px\cos qx\,dx = 0$ if $p - q$ is even;

$= \frac{2p}{p^2-q^2}$ if $p - q$ is odd

456. $\int_0^\infty \frac{\sin ax\sin bx}{x}\,dx = \frac{1}{2}\log\left|\frac{a+b}{a-b}\right| \qquad (a \neq b)$

457. $\int_0^\infty \frac{\sin ax\cos bx}{x}\,dx = 0$ if $a < b$;

$= \frac{\pi}{2}$ if $a > b$;

$= \frac{\pi}{4}$ if $a = b$

458. $\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a}$

459. $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab}$

460. $\int_0^\infty \frac{\sin ax\sin bx}{x^2}\,dx = \frac{\pi a}{2} \qquad (a < b)$

461. $\int_0^{\frac{\pi}{2}} \sin^p x\,dx = \int_0^{\frac{\pi}{2}} \cos^p x\,dx;$

$= \frac{1\cdot 3\cdot 5\cdots(p-1)}{2\cdot 4\cdot 6\cdots p}\cdot\frac{\pi}{2}$ if p is even;

$= \frac{2\cdot 4\cdot 6\cdots(p-1)}{1\cdot 3\cdot 5\cdots p}$ if p is odd

462. $\int_0^{\frac{\pi}{2}} \sin^{p-1}x\cos^{q-1}x\,dx = \frac{1}{2}B\left(\frac{p}{2}, \frac{q}{2}\right)$

463. $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$

464. $\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$

465. $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{\sqrt{\pi}}{2}e^{-2a}$

466. $\int_0^\infty e^{-m\left(\frac{x}{a}-\frac{b}{x}\right)^2}\,dx = \frac{a}{2}\sqrt{\frac{\pi}{m}}$

467. $\int_0^\infty x^{2p}e^{-a^2x^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2p-1)\sqrt{\pi}}{2^{p+1}a^{2p+1}}$

468. $\int_0^\infty x^{2p+1}e^{-a^2x^2}\,dx = \frac{p!}{2a^{2p+2}}$

469. $\int_0^\infty e^{-ax}\sin kx\,dx = \frac{k}{a^2+k^2}$

470. $\int_0^\infty e^{-ax}\cos kx\,dx = \frac{a}{a^2+k^2}$

471. $\int_0^\infty xe^{-ax}\sin kx\,dx = \frac{2ak}{(a^2+k^2)^2}$

472. $\int_0^\infty xe^{-ax}\cos kx\,dx = \frac{a^2-k^2}{(a^2+k^2)^2}$

473. $\int_0^\infty \frac{e^{-ax}\sin kx\,dx}{x} = \arctan\frac{k}{a}$

474. $\int_0^\infty e^{-a^2x^2}\cos kx\,dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{k^2}{4a^2}}$

475. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}$

476. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}$

477. $\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}$

478. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\log\frac{1}{2}$

479. $\int_0^1 \frac{x^{m-1}-x^{n-1}}{\log x}\,dx = \log\frac{m}{n}$

480. $\int_0^{\frac{\pi}{2}} \log\sin x\,dx = \int_0^{\frac{\pi}{2}} \log\cos x\,dx = \frac{\pi}{2}\log\frac{1}{2}$

481. $\int_0^{\frac{\pi}{2}} \log\tan x\,dx = \int_0^{\frac{\pi}{2}} \log\operatorname{ctn} x\,dx = 0$

482. $\int_0^{\frac{\pi}{2}} \log\sec x\,dx = \int_0^{\frac{\pi}{2}} \log\csc x\,dx = \frac{\pi}{2}\log 2$

483. $\int_0^{\pi} x\log\sin x\,dx = \frac{\pi^2}{2}\log\frac{1}{2}$

484. $\int_0^{\frac{\pi}{2}} \sin x\log\sin x\,dx = \log 2 - 1$

485. $\int_0^1 x^p\log x\,dx = -\frac{1}{(p+1)^2}$

486. $\int_0^1 \frac{x^p\,dx}{\log x} = \log(p+1)$

487. $\int_0^1 \frac{\log(1+x)}{x}\,dx = \frac{\pi^2}{12}$

488. $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$

489. $\int_0^\infty \frac{\log(1+a^2x^2)}{b^2+x^2}\,dx = \frac{\pi}{b}\log(1+ab)$

490. $\int_0^1 \left(\frac{\log x}{x-1}\right)^2 dx = \frac{\pi^2}{3}$

491. $\int_0^\infty \left(\frac{\log x}{x-1}\right)^2 dx = \frac{2\pi^2}{3}$

492. $\int_0^1 \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{16}$

493. $\int_0^\infty \frac{dx}{\sinh ax} = \frac{\pi}{2a}$

494. $\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}$

15 Series

In the following formulas, a, b, d, n, r are constants, all arguments of the trigonometric functions are in radians, and all logarithms are in the natural system.

FINITE SERIES

1. $a + (a + d) + (a + 2d) + \cdots + (a + \overline{n-1}\, d) = \frac{n}{2}[2a + (n-1)d]$

2. $a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right)$

3. $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$

4. $1 + 3 + 5 + \cdots + (2n-1) = n^2$

5. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$

6. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

7. $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n}{30}(n+1)(2n+1)(3n^2 + 3n - 1)$

INFINITE SERIES

8. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$

9. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

10. $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \cdots = \int_0^1 \frac{x^{a-1}\,dx}{1+x^b}$ $\quad (a, b > 0)$

11. $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ $\quad (r^2 < 1)$

12. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

13. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

14. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

15. $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$

16. $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \frac{1}{e}$

POWER SERIES

The interval of convergence is given in parentheses.

17. $$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots \quad \text{(Taylor's series)}$$

If $f(x)$ and its derivatives are continuous at $x = a$, and $\lambda = \lim_{n\to\infty}\left|\frac{nf^{(n-1)}(a)}{f^{(n)}(a)}\right|$, the series converges for $a - \lambda < x < a + \lambda$. In order that the series converge to $f(x)$, it is necessary and sufficient that $\lim_{n\to\infty}\frac{(x-a)^n}{n!}f^{(n)}(x_1) = 0, \quad a < x_1 < x$.

18. $$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \quad \text{(Maclaurin's series)}$$

This series is obtained from Taylor's series when $a = 0$.

19. $$(a+x)^n = a^n + \frac{n}{1!}a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots \quad (x^2 < a^2)$$

This series reduces to a polynomial, valid for any finite value of x, when n is a positive integer.

20. $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad (x^2 < 1)$$

See the note under Series 19.

21. $(1+x)^{-n}=1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\cdots \qquad (x^2<1)$

22. $(1+x)^{-1}=1-x+x^2-x^3+\cdots \qquad (x^2<1)$

23. $(1+x)^{-2}=1-2x+3x^2-4x^3+\cdots \qquad (x^2<1)$

24. $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1\cdot 1}{2\cdot 4}x^2+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4+\cdots \qquad (x^2\leqq 1)$

25. $(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\cdots \qquad (x^2<1)$

26. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \qquad (x^2<\infty)$

27. $a^x=1+\frac{x\log a}{1!}+\frac{(x\log a)^2}{2!}+\frac{(x\log a)^3}{3!}+\cdots \qquad (x^2<\infty)$

28. $\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\frac{B_3}{6!}x^6-\cdots \qquad (x^2<4\pi^2)$

The Bernoulli Numbers:

$$B_1=\frac{1}{6} \qquad B_4=\frac{1}{30} \qquad B_7=\frac{7}{6}$$

$$B_2=\frac{1}{30} \qquad B_5=\frac{5}{66} \qquad B_8=\frac{3617}{510}$$

$$B_3=\frac{1}{42} \qquad B_6=\frac{691}{2730} \qquad B_9=\frac{43{,}867}{798}, \text{ etc.}$$

29. $\log x=\frac{x-1}{x}+\frac{1}{2}\left(\frac{x-1}{x}\right)^2+\frac{1}{3}\left(\frac{x-1}{x}\right)^3+\cdots \qquad \left(x>\frac{1}{2}\right)$

30. $\log x=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots \qquad (0<x\leqq 2)$

31. $\log x=2\left[\frac{x-1}{x+1}+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5+\cdots\right] \qquad (x>0)$

32. $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \qquad (-1<x\leqq 1)$

33. $\log\left(\frac{x+1}{x-1}\right)=2\left[\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}+\cdots\right] \qquad (x^2>1)$

34. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \qquad (x^2<\infty)$

35. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \qquad (x^2<\infty)$

36. $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots+\frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}x^{2n-1}+\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$

37. $\operatorname{ctn} x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\frac{x^7}{4725}-\cdots \qquad (x^2<\pi^2)$

38. $\sec x=1+\frac{x^2}{2}+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8+\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$

39. $\csc x=\frac{1}{x}+\frac{x}{6}+\frac{7}{360}x^3+\frac{31}{15{,}120}x^5+\frac{127}{604{,}800}x^7+\cdots \qquad (x^2<\pi^2)$

40. $\arcsin x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\cdots \qquad (x^2<1)$

41. $\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=\frac{\pi}{2}-\operatorname{arcctn} x \qquad (x^2<1)$

42. $\operatorname{arcctn} x=\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\frac{1}{7x^7}+\cdots=\frac{\pi}{2}-\arctan x \qquad (x^2>1)$

43. $\log|\sin x|=\log|x|-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\cdots-\frac{2^{2n-1}B_n x^{2n}}{n(2n)!}-\cdots \qquad (x^2<\pi^2)$

44. $\log\cos x=-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\frac{17x^8}{2520}-\cdots-\frac{2^{2n-1}(2^{2n}-1)B_n}{n(2n)!}x^{2n}-\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$

45. $\log|\tan x|=\log|x|+\frac{x^2}{3}+\frac{7x^4}{90}+\frac{62x^6}{2835}+\cdots+\frac{2^{2n}(2^{2n-1}-1)B_n}{n(2n)!}x^{2n}+\cdots \qquad \left(x^2<\frac{\pi^2}{4}\right)$

46. $\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots \qquad (x^2<\infty)$

47. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad (x^2 < \infty)$

48. $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315}$

$$+ \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n}{(2n)!} x^{2n-1} + \cdots \qquad \left(x^2 < \frac{\pi^2}{4}\right)$$

49. $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots = \frac{x}{2} \qquad (-\pi < x < \pi)$

50. $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots = \frac{\pi - x}{2}$

$$(0 < x < 2\pi)$$

51. $\cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x - \frac{1}{4}\cos 4x + \cdots = \log 2\cos\frac{1}{2}x$

$$(-\pi < x < \pi)$$

52. $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \frac{1}{4}\cos 4x + \cdots = -\log 2\sin\frac{1}{2}x$

$$(0 < x < 2\pi)$$

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